JOURNAL OF ALGEBRA **197,** 92–126 (1997)
ARTICLE NO. JA977131

Structure of Projectively Stable Artinian Rings

Shashidhar Jagadeeshan*

Centre for Learning, 462, 9th Cross, Jayanagar 1st Block, Bangalore 560 011, India

and

Mark Kleiner†

Department of Mathematics, Syracuse University, Syracuse, New York 13244-1150

Communicated by Kent R. Fuller

Received March 22, 1996

1. INTRODUCTION

In this paper, all rings are associative with $1 \neq 0$ and, unless indicated otherwise, all modules are unital left modules. We use freely the definitions, notation, and results from [AF] on rings and modules.

For a semiperfect ring Λ with radical $\mathbf{r}$, $\operatorname{Mod}\Lambda$ is the category of Λ-modules, $\operatorname{mod}\Lambda$ is the category of finitely presented Λ-modules, and $\operatorname{mod}_P \Lambda$ is the full subcategory of $\operatorname{mod}\Lambda$ determined by the modules without nonzero projective direct summands. If $M \in \operatorname{mod}\Lambda$ is of finite length, $l(M)$ is the length of M. Given a two-sided ideal I of Λ, we identify the category $\operatorname{Mod}(\Lambda/I)$ with the full subcategory of $\operatorname{Mod}\Lambda$ determined by those modules M that satisfy $IM = 0$.

DEFINITION 1.1. A semiperfect ring Λ is said to be (left) *projectively stable* if, for all M and N in $\operatorname{mod}_P \Lambda$, no nonzero morphism f: $M \to N$ factors through a projective.

We say "projectively stable" instead of "stable" as in our paper [JK1], to avoid confusion: it has come to our attention that there are at least two

* E-mail address: shashi@cfl.ilban.ernet.in.
† E-mail address: mkleiner@sound.syr.edu.

0021-8693/97 $25.00

distinct classes of rings called stable. Our use of the word "stable" differs from that in [SV] and also from that in [DP]. These differences will be spelled out in Section 2.

Of course, there is a definition dual to Definition 1.1. Denote by $\operatorname{mod}_I \Lambda$ the full subcategory of $\operatorname{mod} \Lambda$ whose objects are all modules without nonzero injective direct summands. A semiperfect ring Λ is (left) *injectively stable* if, for all M and N in $\operatorname{mod}_I \Lambda$, no nonzero morphism f: $M \to N$ factors through an injective. If Λ is an artin algebra, the duality D: $\operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$ [ARS] allows one to reduce the injectively stable case to projectively stable. Namely, Λ is injectively stable if and only if Λ^{op} is projectively stable. The study of injectively stable rings in general seems an interesting project but it is beyond the scope of this paper.

In what follows, unless indicated otherwise, the word "stable" will always mean "left projectively stable."

There are several reasons to study stable rings. One is that the category $\operatorname{mod} \Lambda$ changes the least when passing to the stable category ($\operatorname{mod} \Lambda$ modulo projectives) if Λ is stable [JK1]. Another has to do with the operation Tr: $\operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$, which is important in representation theory, especially in the theory of almost split sequences over an artin algebra. The operation is well defined on objects but not on morphisms [A, pp. 27–28] because if a morphism f: $M \to N$ in $\operatorname{mod} \Lambda$ and minimal projective presentations $P_1(M) \xrightarrow{g_1} P_0(M) \xrightarrow{g_0} M \to 0$ and $P_1(N) \xrightarrow{h_1} P_0(N) \xrightarrow{h_0} N \to 0$ are given, then morphisms f_0: $P_0(M) \to P_0(N)$, f_1: $P_1(M) \to P_1(N)$ satisfying $fg_0 = h_0 f_0$ and $f_0 g_1 = h_1 f_1$ exist but are not unique. Hence, the morphism $\hat{f}$: $\operatorname{Tr} N = \operatorname{Coker} h_1^* \to \operatorname{Coker} g_1^* = \operatorname{Tr} M$ satisfying $\hat{f} \circ (\operatorname{coker} h_1^*) = (\operatorname{coker} g_1^*) \circ f_1^*$, where $* = \operatorname{Hom}_\Lambda(\ ,\Lambda)$, depends not only on f but also on f_0 and f_1. It was shown in [JK1, Proposition 2.2] for a semiperfect ring Λ that $\hat{f}$ does not depend on f_0 and f_1, or, equivalently, the formula $\operatorname{Tr} f = \hat{f}$ defines a contravariant functor, if and only if Λ^{op} is stable. Note that the result was stated for artin algebras, but the proof works for an arbitrary semiperfect ring. As a continuation of this line of thought, we will characterize in [JK2] all artin algebras Λ for which both operations Tr: $\operatorname{mod} \Lambda \to \operatorname{mod} \Lambda^{\mathrm{op}}$ and Tr: $\operatorname{mod} \Lambda^{\mathrm{op}} \to \operatorname{mod} \Lambda$ are well defined on morphisms and thus are contravariant functors.

Still another reason to study stable rings is that they are a natural generalization of left hereditary rings, which raises the question of how close stable rings are to left hereditary rings. The global dimension of a stable left artinian ring was computed in [JK1, Theorem 4.3], where again the result was stated for artin algebras but the proof works for an arbitrary left artinian ring. As a consequence [JK1, Example 4.1], it was shown that, homologically, stable rings are far away from left hereditary rings in that the global dimension of the latter does not exceed 1 while the former may have arbitrary global dimension, including ∞. On the other hand, the ring structure of stable rings is close to that of left hereditary rings in that, as we will show in Section 3, a large class of stable left artinian rings consists of pullbacks of left hereditary and certain serial rings over semisimple rings.

The three classes of rings that are involved in the above operation of pullback, left hereditary, serial, and semisimple are some of the very few classes of rings for which, in the case when the rings are artin algebras, the structure and representation theory are well understood. Moreover, the artin algebras in each of the three classes can be characterized by a certain property of mod Λ. This brings us to still another reason to study stable rings, as part of the following general problem of representation theory: to describe the structure and representation theory of all artin algebras Λ for which mod Λ has a given property. In the case of stable rings, the property of mod Λ is stated in Definition 1.1 and the ring structure is described in Section 3. The representation theory and the description of quivers with relations for stable artin algebras will be addressed in [JK2], where we will show that the representation theory of stable artin algebras reduces to the representation theory of hereditary and certain serial algebras, including the criteria for finiteness or tameness of representation type and the structure of Auslander−Reiten quiver.

The original version of this paper dealt only with artin algebras, which provided the motivation and main examples. Later the second author realized that most of the proofs can be carried through for arbitrary left artinian rings. In the rest of the paper, all rings are left artinian rings unless indicated otherwise. The absence of the duality D makes some proofs longer but, hopefully, more suitable for a possible generalization to other classes of left artinian rings. We indicate some of these classes below.

In Section 2 we characterize stable rings in terms of their projective modules. For this, the following conditions that a ring may or may not satisfy are useful. We remind the reader that a module is said to be torsionless if it is isomorphic to a submodule of a projective module; a module is said to be cotorsionless if it is isomorphic to a factor module of an injective module.

(A) Every submodule of an indecomposable projective module is either projective or simple.

(B) The injective envelope of each simple nonprojective torsionless module is projective.

(C) The injective envelope of each simple nonprojective torsionless module is uniserial.

(D) Every indecomposable submodule of an indecomposable projective module is either projective or simple.

(E) Every simple nonprojective torsionless module is cotorsionless.

The main result of Section 2 is that a ring is stable if and only if it satisfies (A) and (B), or (A) and (C), or (B) and (D). We also show that a stable ring satisfies (D) and (E) which, in light of [AR, p. 40], means that a stable artin algebra is stably equivalent to an hereditary artin algebra. It would be interesting to know if (D) and (E) characterize rings that are stably equivalent to a left hereditary left artinian ring. In Section 2 we also characterize the intersections of the class of stable rings with the classes of commutative rings, piecewise domains, 1-Gorenstein rings, and serial rings.

The class of left hereditary rings satisfies all five of the above conditions so that looking at any of the five conditions separately gives a class of rings more general than the class of left hereditary rings. Rings satisfying (A) play an especially important role in this paper: most of the results are first obtained for these rings and then specialized to stable rings. In particular, we show in Section 3 that an indecomposable ring satisfying (A) is either left serial with certain restrictions on the lengths of indecomposable projective modules in terms of the quiver of the ring, or a pullback of a left hereditary ring and a certain left serial ring over a semisimple ring. Replacing in the preceding statement the words ‘‘left serial’’ with the word ‘‘serial’’ gives the corresponding result for stable rings. The rings satisfying (A) constitute a much larger class than that of stable rings (e.g., an artin algebra satisfying (A) need not be stably equivalent to an hereditary artin algebra), but our results on the structure of rings satisfying (A) are similar to our results on the structure of stable rings. On the other hand, the representation theory we have [JK2] applies to stable artin algebras only; it would be interesting to extend it to artin algebras satisfying (A).

The techniques of dealing with pullbacks developed in Section 3 may be useful elsewhere. Some of the results presented in the paper were obtained in [J].

2. PROJECTIVE MODULES OVER STABLE RINGS

In this section, unless indicated otherwise, all rings are left artinian and Λ is a ring.

We study properties of morphisms between indecomposable projective modules over a stable ring and use the properties to obtain a characterization of stable rings in terms of their indecomposable projective modules. We compare the class of stable rings to some related classes of rings that have been intensively studied, in particular, in the situation when these rings are artin algebras. Among those classes are rings that are stably equivalent to a left hereditary ring, as well as commutative rings, piecewise domains, 1-Gorenstein, and serial rings. We use the results of this section in Section 3 to describe the ring structure of stable rings.

PROPOSITION 2.1. *Let P be an indecomposable projective Λ-module such that every submodule of P is either projective or simple. Then*:

(a) *P is uniserial if and only if* $\operatorname{Soc} P$ *is simple*.

(b) *If P is not hereditary, it is uniserial with composition series of the form* $0 \subsetneq \operatorname{Soc} P \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_l = P$, *where* $\operatorname{Soc} P$ *is nonprojective simple and, for all j, Q_j is indecomposable projective*.

Proof. (a) The necessity is clear. If $\operatorname{Soc} P$ is simple, it is the unique minimal submodule of P. Therefore, it is enough to show that if P_1, P_2 are nonzero nonsimple submodules of P, then either $P_1 \subset P_2$ or $P_2 \subset P_1$. Modules $P_1, P_2, P_1 + P_2$ must be indecomposable projective. The natural epimorphism $P_1 \oplus P_2 \to P_1 + P_2 \to 0$ splits so that, by the Krull–Remak–Schmidt theorem, either $P_1 + P_2 \cong P_1$ or $P_1 + P_2 \cong P_2$. Hence either $P_1 \subset P_2$ or $P_2 \subset P_1$.

(b) Since P is not hereditary, it has a nonprojective simple submodule S, which is a direct summand of $\operatorname{Soc} P$. It follows that $\operatorname{Soc} P = S$ so that P is uniserial by (a). The rest is clear by assumption. ∎

COROLLARY 2.2. *Assume Λ satisfies* (A) *and let f: $X \to P$ be a morphism of indecomposable modules $X, P \in$* $\operatorname{Mod} \Lambda$, *where P is projective*.

(*a*) *If $0 \neq f$ is not a monomorphism, then P is not hereditary and* $\operatorname{Im} f = \operatorname{Soc} P$. *Thus* $\operatorname{Hom}_\Lambda(X, P) \neq 0$ *if and only if either X is isomorphic to a submodule of P, or P is not hereditary and* $\operatorname{Soc} P$ *is isomorphic to a submodule of $X/\mathbf{r}X$*.

(b) *Suppose P is hereditary and X is not isomorphic to a submodule of P. Then $f = 0$ so that* $\operatorname{Hom}_\Lambda(X, P) = 0$.

(c) *Suppose P is not hereditary and X is not isomorphic to a submodule of P. Then* $\operatorname{Hom}_\Lambda(X, P) = \operatorname{Hom}_\Lambda(X, \operatorname{Soc} P)$.

(d) *Assume X is projective and $X/\mathbf{r}X$ is a torsion module. If $f \neq 0$, then f is a monomorphism.*

Proof. (a) Since $\operatorname{Ker} f \neq 0$ and X is indecomposable, $\operatorname{Im} f$ is not projective and P is not hereditary. The rest follows immediately from Proposition 2.1(b).

(b) If $f \neq 0$, (a) implies f is monic so that $X \cong \operatorname{Im} f$, a contradiction.

(c) Clearly, $\operatorname{Hom}_\Lambda(X, \operatorname{Soc} P) \subset \operatorname{Hom}_\Lambda(X, P)$. Since no nonzero morphism $X \to P$ is monic, the reverse inclusion follows from (a).

(d) If f is not monic, (a) implies $X/\mathbf{r}X \cong \operatorname{Soc} P$, a contradiction. ∎

LEMMA 2.3. *Let S be a proper simple submodule of an indecomposable module $M \in \operatorname{Mod} \Lambda$ and let $I(S)$ be an injective envelope of S. If every submodule of $I(S)$ is either projective or simple, M is isomorphic to a projective submodule of $I(S)$.*

Proof. Denote by $i\colon S \to M$ and $j\colon S \to I(S)$ the natural inclusions and by $k\colon M \to I(S)$, a (nonzero) morphism satisfying $j = ki$. We claim $\operatorname{Im} k$ is not simple. If it is, then $\operatorname{Im} k = S$ and the composite of the morphisms $S \xrightarrow{i} M \xrightarrow{k} S$ is an isomorphism, which is impossible because $M \neq S$ is indecomposable. Then $\operatorname{Im} k$ is projective so that k must be monic because M is indecomposable. ∎

We now present the main result of this section.

THEOREM 2.4. *The following are equivalent for a left artinian ring Λ:*

(a) *Λ is stable.*

(b) *Λ satisfies* (A) *and* (B).

(c) *Λ satisfies* (B) *and* (D).

(d) *Λ satisfies* (A) *and* (C).

Proof. (a) ⇒ (b) Let P be an indecomposable projective Λ-module. If M is a submodule of P that is neither projective nor simple, then $M = M' \oplus Q$, where Q is projective and $0 \neq M' \in \operatorname{mod}_P \Lambda$. We claim M has a submodule L with $0 \neq L \neq M$ such that the composite of the natural inclusion $M' \to P$ and the natural projection $P \to P/L$ is not zero. To prove the claim, consider the cases when M' is or is not simple. If M' is not simple, choose L to be any proper nonzero submodule of M'. If M' is simple, then $Q \neq 0$ and we choose L to be any submodule of Q. Since P is indecomposable, $P/L \in \operatorname{mod}_P \Lambda$, in contradiction with Λ being stable. Thus Λ satisfies (A).

Let S be a nonprojective simple submodule of P. Since Λ satisfies (A), Proposition 2.1(b) implies $S = \operatorname{Soc} P$, whence P is a submodule of the injective envelope $I(S)$ of S. To show Λ satisfies (B), we prove $I(S)$ is projective. If not, take $P \subset Y \subset I(S)$, Y of finite length. Since Y is indecomposable, we can choose Y nonprojective. The inclusions $S \subset P \subset Y$ are impossible since Λ is stable.

(b) ⇒ (a) Suppose $M, N \in \operatorname{mod}_P \Lambda$ and let $f\colon M \to N$ satisfy $f = kh$, where $h\colon M \to P$, $k\colon P \to N$, and P is projective. To show $f = 0$, it suffices to consider the case when $h \neq 0$, $k \neq 0$, and P is indecomposable. Since

$M \in \operatorname{mod}_P \Lambda$, $\operatorname{Im} h$ is not projective, whence P is not simple. By (A), $\operatorname{Im} h$ is simple so that $\operatorname{Im} h = \operatorname{Soc} P$ by Proposition 2.1(b). Since $\operatorname{Soc} P$ is the least nonzero submodule of P, it is enough to show $\operatorname{Ker} k \neq 0$. If $\operatorname{Ker} k = 0$, then k is monic, whence for some indecomposable direct summand N_1 of N, the induced morphism $k_1\colon P \to N_1$ is monic. Since P is not simple, $k_1(\operatorname{Soc} P) \cong \operatorname{Soc} P$ is a proper submodule of N_1. By (B), the injective envelope of $\operatorname{Soc} P$ is projective. Since Λ satisfies (A), Lemma 2.3 implies N_1 is projective, in contradiction with $N \in \operatorname{mod}_P \Lambda$.

(b) $\Rightarrow$ (c) This is trivial.

(c) $\Rightarrow$ (d) By (D), an indecomposable nonhereditary projective Λ-module P contains a nonprojective simple submodule S. By (B), the injective envelope $I(S)$ of S is projective. Since every nonzero submodule of $I(S)$ is indecomposable, (D) implies that every submodule of $I(S)$ is either projective or simple. By Lemma 2.3, P is isomorphic to a submodule of $I(S)$, so that (A) holds. Now Proposition 2.1(a) and (B) imply that Λ satisfies (C).

(d) $\Rightarrow$ (b) We show Λ satisfies (B). Let S be a nonprojective simple submodule of an indecomposable projective Λ-module P; (A) and Proposition 2.1(b) imply $S = \operatorname{Soc} P$. Let I be the injective envelope of S and P; denote by $i\colon P \to I$ the natural inclusion. We prove that a projective cover $\pi\colon P(I) \to I$ of I is an isomorphism. By (C), I is uniserial, so that $P(I)$ is indecomposable. Since P is projective, $i = \pi k$ for some $k\colon P \to P(I)$, where k is monic. Then $\operatorname{Soc} P(I) \cong S$ by Proposition 2.1(b), I is an injective envelope of $P(I)$, and π is an isomorphism. ∎

For examples of stable rings, we refer the reader to Corollary 2.21, which gives a description of stable serial rings, and to Section 3. In [JK2] we will describe the path algebras of quivers with relations that are stable finite-dimensional algebras.

We establish some useful properties of a projective cover of a nonprojective simple torsionless module over a stable ring. An indecomposable hereditary projective module P is said to be *maximal hereditary* if every nonzero morphism $P \to Q$ into an indecomposable hereditary projective module Q is an isomorphism.

PROPOSITION 2.5. *Assume Λ is stable. Let $\pi\colon P \to S$ be a projective cover of a nonprojective simple torsionless Λ-module S.*

(a) *π factors through an indecomposable injective Λ-module* (*not necessarily of finite length*); *moreover, S is cotorsionless and not injective.*

(b) *If P is not hereditary, it is injective. If P is hereditary, it is maximal hereditary.*

Proof. Let $i\colon S \to I(S)$ and $j\colon P \to I(P)$ be injective envelopes of S and P, respectively.

(a) Let $k\colon I(P) \to I(S)$ be a morphism satisfying $i\pi = kj$. Since Λ is left artinian, $I(P)$ is a finite direct sum of indecomposable injective modules [AF, Ex. 18.31]. Since $i\pi \neq 0$, there exists an indecomposable summand E of $I(P)$ for which the composite of the morphisms $j_1\colon P \to E$ and $k_1\colon E \to I(S)$, induced by j and k, respectively, is not zero. It suffices to show that $\operatorname{Im} k_1 = S$, for then $k_1 j_1\colon P \to S$ is a projective cover.

Suppose, to the contrary, that $\operatorname{Im} k_1 \neq S$. Since $I(S)$ is indecomposable projective by (B) and S is a unique simple submodule of $I(S)$, then $\operatorname{Im} k_1$ is a projective submodule of $I(S)$ according to (A). Since E is indecomposable, $\operatorname{Ker} k_1 = 0$ whence $\operatorname{Im} k_1 = I(S)$ because $I(S)$ is indecomposable. Thus, k_1 is an isomorphism, whence S is a submodule of $\operatorname{Soc} P$. By Proposition 2.1(b), $S = \operatorname{Soc} P$ so that $E = I(P)$ and k is an isomorphism. Then $\operatorname{Ker}(kj) = 0$, while $0 \neq \operatorname{Ker}(i\pi) = \operatorname{Ker} \pi$, in contradiction with $i\pi = kj$.

The remaining statement of part (a) is clear.

(b) If P is not hereditary, Proposition 2.1(b) implies $\operatorname{Soc} P$ is simple not projective so that $I(P)$ is indecomposable and, by (B), projective. By (a), there exists a morphism $h\colon I(P) \to S$ satisfying $\pi = hj$. Hence $\operatorname{Im} j \not\subset \mathbf{r}I(P)$ and j is onto.

If P is hereditary and $0 \neq u\colon P \to Q$ is a morphism with Q indecomposable hereditary projective, then u is monic and, by (a), $\pi = hu$ for some $h\colon Q \to S$. Then $\operatorname{Im} u \not\subset \mathbf{r}Q$ and u is onto. ▮

COROLLARY 2.6. (a) *A stable ring satisfies* (D) *and* (E).

(b) *A stable artin algebra is stably equivalent to an hereditary artin algebra.*

Proof. (a) Follows from Theorem 2.4 and Proposition 2.5(a).

(b) Follows from (a) because (D) and (E) characterize artin algebras stably equivalent to an hereditary artin algebra [AR, p. 40]. ▮

It would be interesting to know whether conditions (D) and (E) characterize left artinian rings that are stably equivalent to a left hereditary ring.

In connection with Corollary 2.6, we give the following simple example.

EXAMPLE 2.7. Consider the quiver $G = \{v(G), a(G)\}$ with the set of vertices $v(G) = \{1, 2, 3, 4\}$ and the set of arrows $a(G) = \{a, b, c\}$, where $a\colon 2 \to 1$, $b\colon 3 \to 2$, $c\colon 4 \to 2$. For a field k, the quotient of the path algebra $k[G]$ modulo the ideal I generated by ab and ac satisfies (A) and (E) but not (B). Hence $k[G]/I$ is stably equivalent to an hereditary artin algebra but is not stable.

This example shows that the class of stable rings is strictly smaller than that of rings satisfying (A) and (E). It also shows that the converse of Corollary 2.6(b) is false, i.e., the class of stable artin algebras is strictly smaller than that of artin algebras stably equivalent to hereditary.

Notation 2.8. For a ring Λ, assume that $\mathsf{T} = \{T_1, \dots, T_w\}$, $w > 0$, is a complete set of pairwise nonisomorphic nonprojective simple torsionless Λ-modules. Denote by v an integer satisfying $0 < v \leq w$ and by $\mathbf{t}$, the two-sided ideal of Λ equal to the sum of the homogeneous components of $T_1, \dots, T_v$ in Soc Λ. For any $M \in \operatorname{Mod}\Lambda$, denote by $\tau_{\mathbf{t}}(M)$ the trace of $\mathbf{t}$ in M [AF, Sect. 8], i.e., the sum of the homogeneous components of $T_1, \dots, T_v$ in Soc M. Under the assumption, $\mathbf{t} \subset \mathbf{r}$ so that $\mathbf{t}^2 = 0$, every indecomposable projective $\Lambda/\mathbf{t}$-module is of the form $P/\mathbf{t}P$ for some indecomposable projective Λ-module P, and if an indecomposable projective Λ-module Q satisfies $P \not\cong Q$, then $P/\mathbf{t}P \not\cong Q/\mathbf{t}Q$.

The reason for assuming $w > 0$ is that if $w = 0$ and Λ satisfies (A) or (D), then Λ is left hereditary, while we are mainly interested in nonhereditary stable rings.

The next result extends [Pl, Lemma 5.1, p. 110, and Lemma 5.5, p. 112] to any associative ring.

LEMMA 2.9. *In the setting of Notation* 2.8, *let* $I(S)$ *be an injective envelope of a simple* Λ*-module* S ($I(S)$ *need not be of finite length*).

(a) *If* P *is a projective* Λ*-module, then* $\mathbf{t}P = \tau_{\mathbf{t}}(P)$.

(b) $\mathbf{t}I(S) = S$ *if* $S \cong T_j$ *for some* $j \leq v$, *and* $\mathbf{t}I(S) = 0$ *otherwise*.

(c) *If* $S \not\cong T_j$ *for all* $j \leq v$, *then* $I(S)$ *is an injective envelope of* S *in* $\operatorname{Mod}(\Lambda/\mathbf{t})$.

Proof. We do not use that the T_js are not projective as in Notation 2.8.

(a) By definition, $\mathbf{t}\Lambda = \tau_{\mathbf{t}}(\Lambda)$. If $M \in \operatorname{Mod}\Lambda$, then $M = \oplus_\alpha M_\alpha$, $\alpha \in A$, implies $\mathbf{t}M = \oplus_\alpha \mathbf{t}M_\alpha$. Since $P \oplus Q = \oplus_\alpha \Lambda$ for some Q, we get by [AF, Proposition 8.18] that $\tau_{\mathbf{t}}(P) \oplus \tau_{\mathbf{t}}(Q) = \oplus_\alpha \tau_{\mathbf{t}}(\Lambda) = \oplus_\alpha \mathbf{t}\Lambda = \mathbf{t}P \oplus \mathbf{t}Q$. Hence $\mathbf{t}P = \tau_{\mathbf{t}}(P)$.

(b) For $M \in \operatorname{Mod}\Lambda$, it is clear that $\mathbf{t}M \subset \tau_{\mathbf{t}}(M) \subset \operatorname{Soc} M$. Since $\operatorname{Soc} I(S) = S$, it is obvious that if $S \not\cong T_j$ for all $j \leq v$, then $\tau_{\mathbf{t}}(I(S)) = 0$, whence $\mathbf{t}I(S) = 0$. If $S \cong T_j$ for some $j \leq v$, then S is a submodule of a projective Λ-module P and, by (a), $S \subset \mathbf{t}P$. Denote by h: $S \to P$ and i: $S \to I(S)$ the natural inclusions, and by k: $P \to I(S)$, a morphism satisfying $i = kh$. Then $S = \operatorname{Im} i = \operatorname{Im} kh \subset k(\mathbf{t}P) \subset \mathbf{t}I(S) \subset \operatorname{Soc} I(S) = S$. Thus $\mathbf{t}I(S) = S$.

(c) By (b), $\mathbf{t}I(S) = 0$ so that $I(S)$ and $S = \operatorname{Soc} I(S)$ are $\Lambda/\mathbf{t}$-modules. Since $\operatorname{Mod}(\Lambda/\mathbf{t})$ is identified with a full subcategory of $\operatorname{Mod}\Lambda$, $I(S)$ must be injective in $\operatorname{Mod}(\Lambda/\mathbf{t})$. ∎

We partition the indecomposable projective $\Lambda/\mathbf{t}$-modules into three groups according to how they are obtained from the indecomposable projective Λ-modules, then show that if Λ satisfies (A) (is stable), then $\Lambda/\mathbf{t}$ satisfies (A) (is stable).

PROPOSITION 2.10. *In the setting of Notation* 2.8, *assume* Λ *satisfies* (A) *and P is an indecomposable projective* Λ-*module*.

(a) *If P is hereditary, then* $\mathbf{t}P = 0$ *and P is an indecomposable hereditary projective* $\Lambda/\mathbf{t}$-*module*.

(b) *If* $\operatorname{Soc} P \cong T_j$ *for some* $j \le v$, *then* $P/\mathbf{t}P$ *is an indecomposable hereditary projective* $\Lambda/\mathbf{t}$-*module*.

(c) *If P is not hereditary and* $\operatorname{Soc} P \not\cong T_j$ *for all* $j \le v$, *we have* $\mathbf{t}P = 0$ *and P is an indecomposable projective* $\Lambda/\mathbf{t}$-*module every submodule of which is either projective or simple*. *Moreover*, *P is hereditary over* $\Lambda/\mathbf{t}$ *if and only if the projective cover Q of* $\operatorname{Soc} P$ *in* $\operatorname{mod}\Lambda$ *is of length* 2 *and* $\operatorname{Soc} Q \cong T_j$ *for some* $j \le v$.

(d) $\Lambda/\mathbf{t}$ *satisfies* (A) *and has at least one indecomposable hereditary projective module*. *Any nonprojective simple torsionless* $\Lambda/\mathbf{t}$-*module is isomorphic to T_j for some j satisfying* $v < j \le w$; *in particular*, *if* $v = w$, *then* $\Lambda/\mathbf{t}$ *is a left hereditary ring*.

(e) *If* Λ *is stable*, $\Lambda/\mathbf{t}$ *is stable and*, *for all* $j \le v$, T_j *is an injective* $\Lambda/\mathbf{t}$-*module*.

Proof. (a) Any submodule Q of P is projective so that $\tau_{\mathbf{t}}(Q) = 0$, whence $\mathbf{t}Q = 0$ by Lemma 2.9(a). Thus every submodule of P is a projective $\Lambda/\mathbf{t}$-module.

(b) By Lemma 2.9(a), $\operatorname{Soc} P = \tau_{\mathbf{t}}(P) = \mathbf{t}P$. By assumption, P is not hereditary so that, by Proposition 2.1(b), every submodule Q of P properly containing $\mathbf{t}P$ is an indecomposable projective Λ-module and $\operatorname{Soc} Q = \operatorname{Soc} P$. By Lemma 2.9(a), $\operatorname{Soc} Q = \mathbf{t}Q = \mathbf{t}P$, whence every nonzero submodule of $P/\mathbf{t}P$ is of the form $Q/\mathbf{t}Q$, i.e., is a projective $\Lambda/\mathbf{t}$-module.

(c) Since $\operatorname{Soc} P$ is simple by Proposition 2.1(b), Lemma 2.9(a) implies $\mathbf{t}P = \tau_{\mathbf{t}}(P) = 0$ so that P is projective over $\Lambda/\mathbf{t}$ and, for all submodules Y of P, we have $\mathbf{t}Y = 0$. Since Λ satisfies (A), Y is either projective or simple over Λ, hence, either projective or simple over $\Lambda/\mathbf{t}$.

We prove the sufficiency of the remaining statement by showing that $\operatorname{Soc} P$ is a projective $\Lambda/\mathbf{t}$-module. Since $l(Q) = 2$, then $\operatorname{Soc} P \cong Q/\operatorname{Soc} Q$ and, by (b), $\operatorname{Soc} P$ is a projective $\Lambda/\mathbf{t}$-module. For the necessity, we are given that $\operatorname{Soc} P$ is projective over $\Lambda/\mathbf{t}$. Then $\mathbf{t}Q \neq 0$ for, otherwise, the projective cover $Q \to \operatorname{Soc} P$ splits in $\operatorname{mod} \Lambda/\mathbf{t}$ and, hence, in $\operatorname{mod} \Lambda$, in contradiction with $\operatorname{Soc} P$ being nonprojective over Λ. By (a), (b), and what we have already proved in (c), $\operatorname{Soc} Q \cong T_j$ for some $j \leq v$. By (b), we obtain a surjection $Q/\mathbf{t}Q \to \operatorname{Soc} P$ of indecomposable projective $\Lambda/\mathbf{t}$-modules which must be an isomorphism. Since $\operatorname{Soc} Q = \mathbf{t}Q$, then $l(Q) = 2$.

(d) Parts (a), (b), and (c) imply the first statement, as well as the fact that a simple $\Lambda/\mathbf{t}$-module has a chance of being nonprojective torsionless only if it is isomorphic to $\operatorname{Soc} Q$, where Q is an indecomposable nonhereditary projective Λ-module with $\operatorname{Soc} Q \not\cong T_j$ for all $j \leq v$. The latter observation implies the second statement.

(e) By Theorem 2.4, Λ satisfies (A) and (B). By (d), $\Lambda/\mathbf{t}$ satisfies (A) so that we have to show $\Lambda/\mathbf{t}$ satisfies (B). Let T be a nonprojective simple torsionless $\Lambda/\mathbf{t}$-module. By (d), T is a nonprojective simple torsionless Λ-module with $T \not\cong T_j$ for all $j \leq v$. By Lemma 2.9(b), an injective envelope I of T in $\operatorname{Mod} \Lambda$ satisfies $\mathbf{t}I = 0$. Therefore, by Lemma 2.9(c), I is an injective envelope of T in $\operatorname{Mod}(\Lambda/\mathbf{t})$. Since Λ satisfies (B), I is projective over Λ, hence, projective over $\Lambda/\mathbf{t}$ because $\mathbf{t}I = 0$. Thus $\Lambda/\mathbf{t}$ satisfies (B), hence, is stable.

By (B), an injective envelope $I(T_j)$ of T_j in $\operatorname{Mod} \Lambda$ is projective. By Proposition 2.1(b), any submodule Q of $I(T_j)$ with $0 \neq Q \neq T_j$ is projective and $T_j = \operatorname{Soc} Q$. An injective envelope $E(T_j)$ of T_j in $\operatorname{Mod}(\Lambda/\mathbf{t})$ is $E(T_j) = \{x \in I(T_j) \mid \mathbf{t}x = 0\}$ by [AF, Ex. 16.14(2)]. Since $\mathbf{t}T_j = 0$ and $\mathbf{t}Q = T_j$ by Lemma 2.9(a), $T_j = E(T_j)$. ∎

Over a left hereditary ring, every factor module of an injective module is injective. We need a weaker version of this property for stable rings.

Recall that a submodule Y of a Λ-module X is fully invariant if $f(Y) \subset Y$ for all Λ-endomorphisms f of X. The following statement holds for any ring Λ.

LEMMA 2.11. *Let U be a fully invariant submodule of an injective Λ-module I. Denote by $\mathcal{U}$, $\mathcal{Y}$, and $\mathcal{Y}/U$ the full subcategories of* $\operatorname{Mod} \Lambda$ *with objects, respectively, the single module U, any set of fully invariant submodules of I that contain U, and all submodules of I/U of the form Y/U with $Y \in \mathcal{Y}$.*

(a) *The map $\Phi\colon \mathcal{Y} \to \mathcal{U}$ given by $\Phi(f) = f|U$ for $f \in \operatorname{Hom}_\Lambda(Y, Z)$, $Y, Z \in \mathcal{Y}$, is an additive functor. For $Y \in \mathcal{Y}$, the map $\Phi(Y, Y)\colon \operatorname{End}_\Lambda(Y) \to \operatorname{End}_\Lambda(U)$ is a surjective homomorphism of rings.*

(b) *The map $\Psi\colon \mathcal{Y} \to \mathcal{Y}/U$ given by $\Phi(Y) = Y/U$ for $Y \in \mathcal{Y}$ and by $\Psi(f) = \bar{f}$ for $f \in \operatorname{Hom}_\Lambda(Y, Z)$, $Y, Z \in \mathcal{Y}$, where $\bar{f}(y + U) = f(y) + U$ for*

$y \in Y$, is an additive functor. The kernel of $\Psi(Y,Z)$: $\mathrm{Hom}_\Lambda(Y,Z) \to \mathrm{Hom}_\Lambda(Y/U, Z/U)$ *is* $\mathrm{Hom}_\Lambda(Y,U)$, *so that* Ψ *is faithful if and only if* $\mathrm{Hom}_\Lambda(Y,U) = 0$ *for* $Y \in \mathsf{Y}$.

Proof. (a) Since I is injective, any morphism $f\colon Y \to Z$ can be extended to an endomorphism $\hat{f}\colon I \to I$ so that $f|U = \hat{f}|U$. Since $\hat{f}|U \in \mathrm{End}_\Lambda(U)$, Φ is well defined. We leave to the reader to check that Φ is an additive functor. If $g \in \mathrm{End}_\Lambda(U)$, then $\hat{g} \in \mathrm{End}_\Lambda(I)$ and $g = \hat{g}|U = (\hat{g}|Y)|U$, whence $\Phi(Y,Y)$ is surjective.

(b) By (a), each $f \in \mathrm{Hom}_\Lambda(Y,Z)$ gives rise to the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \longrightarrow & Y & \longrightarrow & Y/U & \longrightarrow & 0 \\ & & {\scriptstyle \Phi(f)}\big\downarrow & & {\scriptstyle f}\big\downarrow & & & & \\ 0 & \longrightarrow & U & \longrightarrow & Z & \longrightarrow & Z/U & \longrightarrow & 0 \end{array}$$

and thus to a unique Λ-morphism $\bar{f}\colon Y/U \to Z/U$ making the diagram commute. Since $\bar{f}(y + U) = f(y) + U$, the map Ψ is well defined. We leave to the reader to check that Ψ is an additive functor. Clearly, $\bar{f} = 0$ if and only if $\mathrm{Im}\, f \subset U$, whence $\mathrm{Ker}\ \Psi(Y,Z) = \mathrm{Hom}_\Lambda(Y,U)$. ∎

Notation 2.12. Assume Λ satisfies (A) and P is an indecomposable projective injective Λ-module. By Proposition 2.1(a) and (b), P is a uniserial module with the composition series $0 \subsetneq \mathrm{Soc}\, P = Q_1 \subsetneq \cdots \subsetneq Q_l = P$ if P is hereditary, or $0 \subsetneq \mathrm{Soc}\, P \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_l = P$ if P is not hereditary; $\{Q_1, \dots, Q_l\}$, $l > 0$, is the set of nonzero projective submodules of P. Denote by $\mathbf{UTM}_l(F)$ ($\mathbf{LTM}_l(F)$) the upper (lower) triangular matrix ring of order l over a ring F.

PROPOSITION 2.13. *In the setting of Notation* 2.12:

(a) $F = \mathrm{End}_\Lambda(\mathrm{Soc}\, P)$ *is a division ring*.

(b) *If* $\mathrm{Soc}\, P \not\cong P/\mathbf{r}P$, *then* $\mathrm{End}_\Lambda(Q_1 \oplus \cdots \oplus Q_l) \cong \mathbf{LTM}_l(F)$.

(c) *If* P *is not hereditary*, $\mathrm{End}_\Lambda((Q_1/\mathrm{Soc}\, P) \oplus \cdots \oplus (Q_l/\mathrm{Soc}\, P)) \cong \mathbf{LTM}_l(F)$.

Proof. (a) Follows from Schur's lemma because $\mathrm{Soc}\, P$ is simple.

(b) Since $\mathrm{Soc}\, P \not\cong P/\mathbf{r}P$, Corollary 2.2(a) implies that every nonzero morphism $P \to P$ is monic. Hence $\mathrm{End}_\Lambda(P)$ is a division ring isomorphic to F by Lemma 2.11(a). Every nonzero morphism $Q_h \to Q_j$ is monic because it can be extended to an automorphism $P \to P$. Hence

$\operatorname{Hom}_\Lambda(Q_h, Q_j) = 0$ if $h > j$, and if $h \le j$, then $\operatorname{Hom}_\Lambda(Q_h, Q_j)$ can be identified with F in a natural way by Lemma 2.11(a) applied for $U = Q_h$. Therefore $\operatorname{End}_\Lambda(Q_1 \oplus \cdots \oplus Q_l) \cong \mathbf{LTM}_l(F)$.

(c) Suppose first that $\operatorname{Soc} P \cong P/\mathbf{r}P$, i.e., P is a projective cover of $\operatorname{Soc} P$. By Corollary 2.2(a), $\operatorname{Hom}_\Lambda(Q_h, Q_j) = 0$ if $j < h < l$, and any nonisomorphism $f\colon P \to P$ satisfies $\operatorname{Im} f = \operatorname{Soc} P$ so that $\operatorname{rad}(\operatorname{End}_\Lambda(P)) = \operatorname{Hom}_\Lambda(P, \operatorname{Soc} P) \neq 0$. Applying Lemma 2.11(a) and (b) for $U = \operatorname{Soc} P$, we obtain that if $h \le j$, then $\operatorname{Hom}_\Lambda(Q_h/\operatorname{Soc} P, Q_j/\operatorname{Soc} P)$ can be naturally identified with F. Hence $\operatorname{End}_\Lambda((Q_1/\operatorname{Soc} P) \oplus \cdots \oplus (Q_l/\operatorname{Soc} P)) \cong \mathbf{LTM}_l(F)$.

If $\operatorname{Soc} P \not\cong P/\mathbf{r}P$, then $l(Q_j) > 1$ for $j \le l$ and (b) imply $\operatorname{Hom}_\Lambda(Q_j, \operatorname{Soc} P) = 0$. Applying Lemma 2.11(b) for $U = \operatorname{Soc} P$, we have $\operatorname{Hom}_\Lambda(Q_h/\operatorname{Soc} P, Q_j/\operatorname{Soc} P) \cong \operatorname{Hom}_\Lambda(Q_h, Q_j)$ for all h, j. The rest follows from (b). ∎

PROPOSITION 2.14. *In the setting of Notation* 2.12, *assume* Λ *is stable*, P *is not simple, and* $\operatorname{Soc} P \cong P/\mathbf{r}P$.

(a) $\Lambda = \Gamma \times \Theta$ *is a direct product of rings, where* Γ *is indecomposable serial stable artinian and* $\{Q_1, \ldots, Q_l = P\}$ *is a complete set of pairwise nonisomorphic indecomposable projective* Γ*-modules*.

(b) *Every torsion factor module of P is injective*.

Proof. Clearly, P is not hereditary.

(a) Let Q be an indecomposable projective Λ-module with $Q \not\cong Q_j$ for $j = 1, \ldots, l$. By Corollary 2.2(c), we have $\operatorname{Hom}_\Lambda(Q, Q_j) = \operatorname{Hom}_\Lambda(Q, \operatorname{Soc} P) = 0$ because the projective cover of $\operatorname{Soc} P$ is $P \not\cong Q$. By Lemma 2.3, any morphism $Q_j \to Q$ is not monic. Since $Q_j/\mathbf{r}Q_j$ is a torsion module for $j < l$ by Proposition 2.5(b), we conclude, based on Corollary 2.2(a) and (d), that $\operatorname{Hom}_\Lambda(Q_j, Q) \neq 0$ for some j would imply $j = l$ and $Q_l = P$ is a projective cover of $\operatorname{Soc} Q$. Since P is both a projective cover and an injective envelope of $\operatorname{Soc} P$, it would follow that P is an injective envelope of Q, a contradiction. Hence $\operatorname{Hom}_\Lambda(Q_j, Q) = 0$ so that $\Lambda = \Gamma \times \Theta$, where Γ is an indecomposable stable artinian ring and $\{Q_1, \ldots, Q_l = P\}$ is a complete set of pairwise nonisomorphic indecomposable projective Γ-modules. We still have to show that every indecomposable injective Γ-module is uniserial [AF, Theorem 32.3], but we first prove (b).

(b) It is easy to see that the homogeneous component $\mathbf{t}$ of $\operatorname{Soc} P$ in $\operatorname{Soc} \Lambda$ is also the homogeneous component of $\operatorname{Soc} P$ in $\operatorname{Soc} \Gamma$. Since hereditary rings and serial rings are preserved by Morita equivalence [AF, Sect. 32] and since, by Proposition 2.13(a) and (c), $\Gamma/\mathbf{t}$ is Morita equivalent to $\mathbf{LTM}_l(F)^{\mathrm{op}} \cong \mathbf{UTM}_l(F)$ with F a division ring, we conclude that $\Gamma/\mathbf{t}$ is an indecomposable serial hereditary artinian ring because so is $\mathbf{UTM}_l(F)$.

Thus $P/\operatorname{Soc} P$ is the unique (up to isomorphism) projective injective $\Gamma/\mathbf{t}$-module, and every factor module of $P/\operatorname{Soc} P$ is an injective $\Gamma/\mathbf{t}$-module. Hence every torsion factor module Z of P is an injective $\Gamma/\mathbf{t}$-module. By Lemma 2.9(c), Z is an injective Γ- and Λ-module, and (b) is proven.

Since $P/\mathbf{r}P$ is the only proper torsionless factor module of P by Proposition 2.5(b), all l indecomposable injective Γ-modules are uniserial. Hence Γ is a serial ring, and the proof of (a) is complete. ∎

PROPOSITION 2.15. *In the setting of Notation* 2.12, *assume* Λ *is stable and* $P/\mathbf{r}P$ *is a torsion* Λ-*module*.

(a) *If* P *is hereditary, then* $\Lambda = \Gamma \times \Delta$ *is a direct product of rings, where* Γ *is indecomposable serial hereditary artinian and* $\{Q_1, \dots, Q_l = P\}$ *is a complete set of pairwise nonisomorphic indecomposable projective* Γ-*modules*.

(b) *If* P *is not hereditary and* $\mathbf{t}$ *is the homogeneous component of* $\operatorname{Soc} P$ *in* $\operatorname{Soc} \Lambda$, *then* $\Lambda/\mathbf{t} = \Sigma \times \Theta$ *is a direct product of rings, where* Σ *is indecomposable serial hereditary artinian and* $\{Q_1/\operatorname{Soc} P, \dots, Q_l/\operatorname{Soc} P = P/\operatorname{Soc} P\}$ *is a complete set of pairwise nonisomorphic indecomposable projective* Σ-*modules*.

(c) *Every factor module of* P *is injective*.

Proof. (a) Let Q be an indecomposable projective Λ-module not isomorphic to a submodule of P. By Corollary 2.2(b), $\operatorname{Hom}_\Lambda(Q, Q_j) = 0$ for all j. Since $Q_j/\mathbf{r}Q_j$ is a torsion module for $1 < j < l$ by Proposition 2.5(b), Corollary 2.2(d) implies that, for $j = 1, \dots, l$, every nonzero morphism $Q_j \to Q$ is monic. If such a morphism exists, $\operatorname{Soc} P$ is a proper submodule of Q so that, in view of Lemma 2.3, Q is isomorphic to a submodule of P, a contradiction. Thus $\operatorname{Hom}_\Lambda(Q_j, Q) = 0$ for all j. It follows that $\Lambda = \Gamma \times \Delta$ and $\{Q_1, \dots, Q_l = P\}$ is a complete set of pairwise nonisomorphic indecomposable projective Γ-modules. Since $\operatorname{Soc} P \not\cong P/\mathbf{r}P$, we finish the proof by using Proposition 2.13(a) and (b) and Morita equivalence.

(b) Let Z be an indecomposable projective $\Lambda/\mathbf{t}$-module not isomorphic to a submodule of $P/\operatorname{Soc} P$. Since P is not hereditary, $Q_j/\operatorname{Soc} P$ is an indecomposable hereditary projective $\Lambda/\mathbf{t}$-module for all j by Proposition 2.10(b). By Corollary 2.2(b), $\operatorname{Hom}_{\Lambda/\mathbf{t}}(Z, Q_j/\operatorname{Soc} P) = 0$. To show $\operatorname{Hom}_{\Lambda/\mathbf{t}}(Q_j/\operatorname{Soc} P, Z) = 0$, we suppose $f\colon Q_j/\operatorname{Soc} P \to Z$ is not zero and arrive at a contradiction.

We claim Z is not a projective Λ-module, for if it is, the composite of f and the natural projection $Q_j \to Q_j/\operatorname{Soc} P$ is not zero and not monic. By Corollary 2.2(a), Q_j is a projective cover of $\operatorname{Soc} Z$ in $\operatorname{mod} \Lambda$, in contradiction with $Q_j/\mathbf{r}Q_j$ being a torsion module by assumption and by Proposition 2.5(b). Then, by Proposition 2.10(a)–(c), $Z \cong Q/\operatorname{Soc} P$, where Q is an indecomposable projective Λ-module and $\operatorname{Soc} Q \cong \operatorname{Soc} P$. Since P is an injective envelope of $\operatorname{Soc} P$ in $\operatorname{Mod} \Lambda$, P is also an injective envelope of Q, so that $Z \cong Q/\operatorname{Soc} P$ is isomorphic to a submodule of $P/\operatorname{Soc} P$, a contradiction. Hence $\operatorname{Hom}_{\Lambda/\mathbf{t}}(Q_j/\operatorname{Soc} P, Z) = 0$.

Thus $\Lambda/\mathbf{t} = \Sigma \times \Theta$ and $\{Q_1/\operatorname{Soc} P, \dots, Q_l/\operatorname{Soc} P = P/\operatorname{Soc} P\}$ is a complete set of pairwise nonisomorphic indecomposable projective Σ-modules. The rest follows from Proposition 2.13(a) and (c) with the use of Morita equivalence.

(c) If P is hereditary, the statement follows from (a) because over an hereditary ring, every factor module of an injective module is injective. If P is not hereditary, a proper factor module Z of P is a factor module of $P/\operatorname{Soc} P$, whence $\operatorname{Soc} Z \cong Q_j/\mathbf{r}Q_j$ for some $j \leq l$. By (b), $P/\operatorname{Soc} P$ is an injective Σ- and $\Lambda/\mathbf{t}$-module, and so is its factor module Z. Note that $\operatorname{Soc} Z \ncong \operatorname{Soc} P$ because $\operatorname{Soc} Z$ is a torsion Λ-module. By Lemma 2.9(c), Z is an injective Λ-module. ∎

If Λ is an artin algebra, the proof of Proposition 2.15(b) can be simplified, using [Pl, Proposition 1.3(a) and (b)] and Corollary 2.6(b).

COROLLARY 2.16. *In the setting of Notation* 2.12, *if* Λ *is stable*, *then every torsion factor module of P is injective*.

Proof. By Propositions 2.14(b) and 2.15(c), we may assume that $S = P/\mathbf{r}P$ is nonprojective torsionless and $\operatorname{Soc} P \ncong S$. Let $\mathbf{s}$ be the homogeneous component of S in $\operatorname{Soc} \Lambda$. By Proposition 2.10(a) and (c), P is an indecomposable projective $\Lambda/\mathbf{s}$-module. By Proposition 2.10(e), $\Lambda/\mathbf{s}$ is stable and S is an injective $\Lambda/\mathbf{s}$-module. It follows that S is a torsion $\Lambda/\mathbf{s}$-module for, otherwise, S would be a projective $\Lambda/\mathbf{s}$-module, in contradiction with the indecomposability and nonsimplicity of P. By Proposition 2.15(c), every factor module Z of P is an injective $\Lambda/\mathbf{s}$-module. If $Z \neq 0$, then either $Z = P/\operatorname{Soc} P$ or $Z = P/Q_j$ for some $j < l$. Clearly, $Z = P/Q_{l-1}$ is a torsionless Λ-module. In the remaining cases, $\operatorname{Soc} Z = Q_h/\mathbf{r}Q_h$, where $h < l$, so that $\operatorname{Soc} Z$ is a torsion Λ-module by Proposition 2.5(b). Then Z is a torsion Λ-module, which is injective by Lemma 2.9(c). ∎

Our next goal is to compare the class of stable rings with several related classes of left artinian rings studied in the literature. Recall [GS] that Λ is a piecewise domain if every nonzero morphism of indecomposable projective Λ-modules is monic. The class of piecewise domains is larger than that of left hereditary rings. Λ is a 1-Gorenstein ring if the injective envelope of a projective Λ-module is projective, or, equivalently, if the injective envelope of a simple torsionless Λ-module is projective. The latter condition is stronger than condition (B), which is one of the two conditions characterizing stable rings. We examine the intersections of the class of stable rings with the classes of commutative rings, piecewise domains, 1-Gorenstein rings, and serial rings.

PROPOSITION 2.17. *The following are equivalent for a commutative artinian ring* Λ:

(a) Λ *is stable*.

(b) Λ *satisfies* (A).

(c) Λ *is serial and* $\mathbf{r}^2 = 0$.

Proof. We may assume Λ local.

(a) $\Rightarrow$ (b) Follows from Theorem 2.4.

(b) $\Rightarrow$ (c) Up to isomorphism, Λ is the only indecomposable projective Λ-module. Since (A) holds and $\mathbf{r} \neq \Lambda$, either $\mathbf{r} = 0$ so that Λ is a field, or $\mathbf{r}$ is simple so that $\mathbf{r}^2 = 0$ and Λ is serial.

(c) $\Rightarrow$ (a) Since $\mathbf{r}^2 = 0$, either $\mathbf{r} = 0$ or $\mathbf{r}$ is a semisimple module which must be simple because Λ is a uniserial module. Hence Λ satisfies (A). Since Λ is serial, it satisfies (C). By Theorem 2.4, Λ is stable. ∎

PROPOSITION 2.18. *The following are equivalent for a left artinian ring* Λ:

(a) Λ *is a stable piecewise domain*.

(b) Λ *is a piecewise domain satisfying* (A).

(c) Λ *is a left hereditary ring*.

Proof. (a) $\Rightarrow$ (b) Follows from Theorem 2.4.

(b) $\Rightarrow$ (c) Over a piecewise domain, a projective cover of a simple torsionless module must be an isomorphism. By Proposition 2.1(b), every indecomposable projective Λ-module is hereditary because its socle is projective.

(c) $\Rightarrow$ (a) Obvious. ∎

PROPOSITION 2.19. *The following are equivalent for a left artinian ring* Λ:

(a) Λ *is a stable* 1-*Gorenstein ring*.

(b) Λ *is a* 1-*Gorenstein ring satisfying* (A).

(c) Λ *is a stable serial ring*.

Proof. (a) $\Leftrightarrow$ (b) Follows from Theorem 2.4: a 1-Gorenstein ring satisfies (B).

(a) $\Rightarrow$ (c) Since Λ is stable, Lemma 2.3 implies that a nonsimple indecomposable projective Λ-module has simple socle, hence, is uniserial by Proposition 2.1(a). We show that an indecomposable injective Λ-module I is uniserial. If $\operatorname{Soc} I$ is torsionless, I is projective because Λ is 1-Gorenstein, so that I is uniserial. If $\operatorname{Soc} I$ is a torsion module, let $\pi: P \to \operatorname{Soc} I$ be a projective cover of $\operatorname{Soc} I$, let $j: P \to E$ be an injective envelope of P, and let $i: \operatorname{Soc} I \to I$ be the natural embedding. There exists a morphism $k: E \to I$ satisfying $i\pi = kj$. Since P is uniserial, E is an indecomposable projective module, hence, uniserial. Since $\operatorname{Soc} I \subset \operatorname{Im} k$, the uniserial module $\operatorname{Im} k$ is a torsion factor module of E. By Corollary 2.16, $\operatorname{Im} k$ is injective so that $I = \operatorname{Im} k$. Thus I is uniserial.

(c) $\Rightarrow$ (a) Let S be a simple torsionless Λ-module. We show that the injective envelope I of S is projective. If S is not projective, this follows from (B) (see Theorem 2.4). If S is projective, I is projective by [B, Lemma 1.2, p. 1273]. ∎

We now mention two classes of rings called stable that appeared in the literature. The first class was defined in [SV], where a commutative ring was called stable if every ideal of the ring is a projective module over its endomorphism ring. The definition extends naturally to the noncommutative case by requiring that every left ideal of a ring be projective over its endomorphism ring. The second class of rings can be defined in several (equivalent) ways; we quote the shortest definition [DP]. A ring Λ is said to be (left) stable if, for each $M \in \operatorname{Mod} \Lambda$ and every injective module $E \in \operatorname{Mod} \Lambda$, $\operatorname{Hom}_\Lambda(M, E) = 0$ implies $\operatorname{Hom}_\Lambda(E(M), E) = 0$, where $E(M)$ is the injective envelope of M. A description of the left artinian rings in this class is given in [P, Theorem 9, pp. 112–113]. Using our description of projectively stable finite-dimensional algebras in terms of quivers with relations [JK2], it is easy to show that neither of the two classes contains the class of projectively stable rings. It seems there are no obvious connections among the three classes.

Finally, we give a characterization of left serial rings satisfying (A) and of serial stable rings. Recall [ARS, Proposition III.1.15, p. 69] that if $\{P_1, \dots, P_n\}$ is a complete set of pairwise nonisomorphic indecomposable projective Λ-modules, the quiver of Λ has the set of vertices $\{1, \dots, n\}$, and it has an arrow $i \to j$ if and only if P_j is a direct summand of the projective cover of $\mathbf{r}P_i$. If Λ is indecomposable serial, the subscripts can be chosen so that P_j is a projective cover of $\mathbf{r}P_{j+1}$ for $j = 1, \dots, n-1$ and if $\mathbf{r}P_1 \neq 0$, then P_n is a projective cover of $\mathbf{r}P_1$. The ordered sequence $\{P_1, \dots, P_n\}$, called a Kupisch series of Λ, is unique up to isomorphism if $\mathbf{r}P_1 = 0$, and up to isomorphism and a cyclic permutation if $\mathbf{r}P_1 \neq 0$ [AF, Theorem 32.4].

PROPOSITION 2.20. *A left serial ring Λ satisfies* (A) *if and only if, for all arrows $i \to j$ of the quiver of Λ, $l(P_j) \geq l(P_i)$ implies $l(P_i) = 2$.*

Proof. Let Λ satisfy (A) and $\mathit{l}(P_j) \geq \mathit{l}(P_i)$ for an arrow $i \to j$. Since $\mathbf{r}P_i \neq P_i$, then $P_j \not\cong \mathbf{r}P_i$ so that $\mathbf{r}P_i$ is not projective and must be simple, whence $\mathit{l}(P_i) = 2$.

For the sufficiency, consider an indecomposable projective Λ-module P with composition series $0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_l = P$. We show that the maximal nonzero nonprojective submodule Z_k of P is simple. A projective cover $Q \to Z_k$ is not monic, and Q and Z_{k+1} are indecomposable projective, whence $\mathit{l}(Q) > \mathit{l}(Z_k) = \mathit{l}(Z_{k+1}) - 1$. For some i and j, we have $Q \cong P_j$ and $Z_{k+1} \cong P_i$ so that, by assumption, $\mathit{l}(P_i) = \mathit{l}(Z_{k+1}) = 2$ and, hence, $\mathit{l}(Z_k) = 1$. ∎

COROLLARY 2.21. *The following are equivalent for an indecomposable serial ring Λ with a Kupisch series $\{P_1, \dots, P_n\}$.*

(a) Λ *is stable.*

(b) Λ *satisfies* (A).

(c) $\mathit{l}(P_j) \geq \mathit{l}(P_{j+1}) > 0$ *implies* $\mathit{l}(P_{j+1}) = 2$ *for* $j = 1, \dots, n$, *where* $P_{n+1} = 0$ *if* $\mathbf{r}P_1 = 0$ *and* $P_{n+1} = P_1$ *if* $\mathbf{r}P_1 \neq 0$.

Proof. (a) ⇒ (b) Follows from Theorem 2.4.

(b) ⇒ (c) Follows from Proposition 2.20.

(c) ⇒ (a) (A) holds by Proposition 2.20; (C) holds because Λ is serial. ∎

3. PULLBACKS AND THE STRUCTURE OF STABLE RINGS

In this section we describe the ring-theoretic structure of stable rings and rings satisfying (A). From now on, we fix a left artinian ring Λ with a complete set $\mathcal{P}$ of cardinality $n > 0$ of pairwise nonisomorphic indecomposable projective Λ-modules and a decomposition of the ring

$$\Lambda = \mathbf{P} \oplus \mathbf{Q}$$

as a direct sum of left ideals $\mathbf{P}$ and $\mathbf{Q}$, where every indecomposable direct summand of $\mathbf{P}$ ($\mathbf{Q}$) is hereditary (nonhereditary). Of course, either $\mathbf{P}$ or $\mathbf{Q}$ may be 0. Since we are mainly interested in stable rings and rings satisfying (A) that are not left hereditary, for the rest of the paper we assume $\mathbf{Q} \neq 0$. Then we have to consider two cases: $\mathbf{P} = 0$ and $\mathbf{P} \neq 0$.

By a chain of length s, we understand a sequence $X_0 - X_1 - \cdots - X_s$ of Λ-modules, where $-$ stands for a nonzero morphism $X_i \to X_{i+1}$ or $X_{i+1} \to X_i$.

LEMMA 3.1. *Assume* Λ *satisfies* (A). *Let* $Q = Q_0 \overset{f_0}{-} Q_1 - \cdots - Q_{s-1} \overset{f_{s-1}}{-} Q_s = P$ *be a chain of minimal length with* P, Q *fixed,* $Q_i \in \mathsf{P}$ *for all* i, *and* $Q_i \ncong Q_j$ *for* $i \neq j$. *Fix an integer* l *satisfying* $0 < l \leq s$.

(a) *If* f_{l-1} *is not a monomorphism and* Q_{l-1} *is the domain of* f_{l-1}, *then* f_l *is not a monomorphism and* Q_l *is the domain of* f_l.

(b) *If* Q_l *is the codomain of both* f_{l-1} *and* f_l, *then* f_{l-1} *and* f_l *are monomorphisms*.

Proof. (a) Since $f_{l-1}\colon Q_{l-1} \to Q_l$ is not monic, Corollary 2.2(a) and Proposition 2.1(b) imply that Q_l is not hereditary, Soc Q_l is nonprojective simple, and Q_{l-1} is a projective cover of Soc Q_l. If, to the contrary, f_l is monic, then Q_{l+1} is the domain of f_l because the chain is of minimal length, whence Soc $Q_{l+1} \cong$ Soc Q_l. The chain $Q = Q_0 - \cdots - Q_{l-1} \overset{g}{-} Q_{l+1} - \cdots - Q_s = P$, where $g\colon Q_{l-1} \to$ Soc Q_{l+1} is a surjective map, is of smaller length than the given one of minimal length, a contradiction. Thus f_l is not monic. The domain of f_l is Q_l, for, otherwise, both Q_{l-1} and Q_{l+1} are projective covers of Soc Q_l, in contradiction with $Q_{l-1} \ncong Q_{l+1}$.

(b) By (a), f_{l-1} is monic. To show f_l is monic, renumber the modules in the chain from right to left and use (a) again. ▮

PROPOSITION 3.2. *Assume* Λ *satisfies* (A) *and let* $Q = Q_0 \overset{f_0}{-} Q_1 - \cdots - Q_{s-1} \overset{f_{s-1}}{-} Q_s = P$ *be a chain with* $Q_i \in \mathsf{P}$ *for all* i *and* $Q_i \ncong Q_j$ *for* $i \neq j$.

(a) *Suppose* $\mathsf{P} = \mathsf{U} \amalg \mathsf{V}$ *is a disjoint union such that, for all* $U \in \mathsf{U}$, $V \in \mathsf{V}$, $\mathrm{Hom}_\Lambda(U, V) = 0$ *and no morphism* $f\colon V \to U$ *is a monomorphism. If* $Q, P \in \mathsf{V}$, *then* $Q_l \in \mathsf{V}$ *for* $l = 1, \ldots, s-1$.

(b) *If* Q *and* P *are hereditary, then* Q_l *is hereditary for* $l = 1, \ldots, s-1$.

(c) *If* Λ *is an indecomposable left serial ring having an indecomposable hereditary projective module, then* Λ *has a unique* (*up to isomorphism*) *simple projective module*.

Proof. (a) Let, to the contrary, the above chain be of minimal length among those chains of nonisomorphic indecomposable projectives with the ends Q and P in which at least one of the terms belongs to U. Let i (j) be the least (largest) positive integer such that Q_i (Q_j) $\in \mathsf{U}$. Then $0 < i \leq j < s$ and Q_{i-1} (Q_{j+1}) $\in \mathsf{V}$. By assumption, Q_{i-1} (Q_{j+1}) is the domain of f_{i-1} (f_j), and f_{i-1} (f_j) is not monic. Since $Q_{i-1} \ncong Q_{j+1}$, Corollary 2.2(a) implies $i \neq j$. Note that Lemma 3.1(b) does not apply here because the chain is not of minimal length among all chains between Q and P. It is easy to see that $Q = Q_0 - Q_1 - \cdots - Q_i - \cdots - Q_{j-1} - Q_j$ is a chain of minimal length of nonisomorphic indecomposable projectives between Q and Q_j. By repeated application of Lemma 3.1(a), beginning with f_{i-1} and f_i, we conclude that f_{j-1} is not monic and Q_{j-1} is the domain of f_{j-1} (remember, $i < j$). Since $Q_i - \cdots - Q_{j-1} - Q_j - Q_{j+1} - \cdots - Q_s = P$ is a chain of minimal length of nonisomorphic indecomposable projectives between Q_i and P, then Q_j is the codomain of f_{j-1} and f_j, in contradiction with Lemma 3.1(b).

(b) By Corollary 2.2(b) and (c), (a) applies if U (V) is a complete set of nonisomorphic indecomposable nonhereditary (hereditary) projective Λ-modules.

(c) By assumption, Λ has at least one simple projective module S. If $T \not\cong S$ is another simple projective, there is a chain $S = P_0 - P_1 - \cdots - P_r = T$ with $P_i \in \mathsf{P}$ for all i and $P_i \not\cong P_j$ for $i \neq j$ because Λ is indecomposable. By (b), P_i is hereditary for all i; since Λ is left serial, P_i is uniserial and $\operatorname{Soc} P_i$ is simple. Since any nonzero morphism of indecomposable hereditary projectives is monic, all P_is have isomorphic socles, in contradiction with $S \not\cong T$. ∎

THEOREM 3.3. *Assume Λ is a left artinian ring.*

(a) *Λ satisfies* (A), $\mathbf{P} = 0$, *and* $\mathbf{Q} \neq 0$ *if and only if it is a left serial ring such that $\mathsf{l}(P) > 1$ for all $P \in \mathsf{P}$ and, for all arrows $i \to j$ of the quiver of Λ, $\mathsf{l}(P_j) \geq \mathsf{l}(P_i)$ implies $\mathsf{l}(P_i) = 2$.*

(b) *Λ is indecomposable stable satisfying* $\mathbf{P} = 0$ *and* $\mathbf{Q} \neq 0$ *if and only if it is a serial ring whose Kupisch series $\{P_1, \dots, P_n\}$ has the following property: for $j = 1, \dots, n$, $\mathsf{l}(P_j) \geq \mathsf{l}(P_{j+1})$ implies $\mathsf{l}(P_{j+1}) = 2$, where $P_{n+1} = P_1$.*

(c) *Assume Λ is indecomposable and satisfies* (A). *If $P \in \mathsf{P}$ is not hereditary and $P/\mathbf{r}P \cong \operatorname{Soc} P$, then $\mathbf{P} = 0$, and $Q/\mathbf{r}Q \cong \operatorname{Soc} Q$ for $Q \in \mathsf{P}$ implies $Q = P$. If Λ is stable, P is, up to isomorphism, the set of distinct nonzero projective submodules of P.*

Proof. (a) If $\mathbf{P} = 0$, every indecomposable projective is not hereditary, hence, nonsimple uniserial by Proposition 2.1(b). Now the necessity follows from Proposition 2.20. If $\mathsf{l}(P) > 1$ for all $P \in \mathsf{P}$, then $\mathbf{P} = 0$, and the sufficiency follows from Proposition 2.20.

(b) If Λ is stable, (A) and (B) hold by Theorem 2.4 so that, by (a), Λ has no simple projectives. By (B), an injective envelope of Λ is projective, Λ is 1-Gorenstein, hence, serial by Proposition 2.19. Now the necessity follows from Corollary 2.21. For the sufficiency, note that the serial ring Λ has no simple projectives, whence $\mathbf{P} = 0$. The rest follows from Corollary 2.21.

(c) Since Λ is indecomposable, if $\mathbf{P} \neq 0$, there exists a chain of minimal length $Q = Q_0 \overset{f_0}{-} \cdots \overset{f_{s-1}}{-} Q_s = P$ with Q hereditary, $Q_i \in \mathsf{P}$ for all i, and $Q_i \not\cong Q_j$ for $i \neq j$. By minimality, Q_1 is not hereditary so that, by Corollary 2.2(b) and (c), Q_0 is the domain of f_0 and a projective cover of $\operatorname{Soc} Q_1$. By Lemma 3.1(a), $f_{s-1}\colon Q_{s-1} \to Q_s$ is not monic so that, by Corollary 2.2(a), Q_{s-1} is a projective cover of $\operatorname{Soc} P$. Since $P/\mathbf{r}P \cong \operatorname{Soc} P$, then $Q_{s-1} \cong P = Q_s$, a contradiction. Thus $\mathbf{P} = 0$ and Λ is left serial by (a), whence each vertex of the quiver of Λ is the source of at most one arrow, and the quiver has a unique full subquiver that is an oriented cycle. Since the full subquiver on the vertices that correspond to the nonzero projective submodules of P (Q) is an oriented cycle, the above uniqueness implies $Q = P$. The case of Λ stable follows from (b). ∎

For the rest of the paper, we assume that $\mathbf{P} \neq 0$ and $\mathbf{Q} \neq 0$. We remind the reader of some facts about pullbacks of rings

$$\begin{array}{ccc} \Lambda & \xrightarrow{k_1} & \Lambda_1 \\ {\scriptstyle k_2}\downarrow & & \downarrow{\scriptstyle l_1} \\ \Lambda_2 & \xrightarrow{l_2} & \overline{\Lambda}. \end{array} \tag{3.1}$$

According to [L, p. 51], all maps in (3.1) are surjective if and only if the diagram (3.1) is isomorphic to one of the form

$$\begin{array}{ccc} \Lambda & \xrightarrow{i_1} & \Lambda/J_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ \Lambda/J_2 & \xrightarrow{j_2} & \Lambda/(J_1 + J_2), \end{array} \tag{3.2}$$

where J_1 and J_2 are two-sided ideals of Λ satisfying $J_1 \cap J_2 = 0$, and the maps are natural projections.

Parts (a) and (b) of the following theorem are proved in [M, Theorem 2.2, p. 20] and [FV, Theorem 1, p. 427], respectively.

THEOREM 3.4. *Let J_1 and J_2 be two-sided ideals of Λ satisfying $J_1 \cap J_2 = 0$.*

(a) *If Q is a projective Λ-module, there exist projective modules Q_1, Q_2, and $\overline{Q}$ over the rings Λ/J_1, Λ/J_2, and $\Lambda/(J_1 + J_2)$, respectively, and* *epimorphisms* $g_i\colon Q \to Q_i$ *and* $h_i\colon Q_i \to \overline{Q}$, *where* $i = 1, 2$, *such that*

$$\begin{array}{ccc} Q & \xrightarrow{g_1} & Q_1 \\ {\scriptstyle g_2}\big\downarrow & & \big\downarrow{\scriptstyle h_1} \\ Q_2 & \xrightarrow{h_2} & \overline{Q} \end{array} \tag{3.3}$$

is a pullback diagram of Λ*-modules.*

(b) *If* E *is an injective* Λ*-module, there exist injective modules* E_1, E_2, *and* $\overline{E}$ *over the rings* Λ/J_1, Λ/J_2, *and* $\Lambda/(J_1 + J_2)$, *respectively, and monomorphisms* $g_i\colon E_i \to E$ *and* $h_i\colon \overline{E} \to E_i$, *where* $i = 1, 2$, *such that*

$$\begin{array}{ccc} E & \xleftarrow{g_1} & E \\ {\scriptstyle g_2}\big\uparrow & & \big\uparrow{\scriptstyle h_1} \\ E_2 & \xleftarrow{h_2} & \overline{E} \end{array}$$

is a pushout diagram of Λ*-modules.*

PROPOSITION 3.5. *Let* J_1 *and* J_2 *be two-sided ideals of* Λ *satisfying* $J_1 \cap J_2 = 0$.

(a) *If every indecomposable projective* $\Lambda/(J_1 + J_2)$*-module is projective over either* Λ/J_1 *or* Λ/J_2, *then* (i) *so is every indecomposable projective* Λ*-module and* (ii) *every indecomposable projective* Λ/J_1*- or* Λ/J_2*-module that is not projective over* $\Lambda/(J_1 + J_2)$ *is projective over* Λ.

(b) *If every indecomposable injective* $\Lambda/(J_1 + J_2)$*-module is injective over either* Λ/J_1 *or* Λ/J_2, *then* (i) *so is every indecomposable injective* Λ*-module and* (ii) *every indecomposable injective* Λ/J_1*- or* Λ/J_2*-module that is not injective over* $\Lambda/(J_1 + J_2)$ *is injective over* Λ.

Proof. (a) An indecomposable projective Λ-module Q comes with the pullback diagram (3.3) of Theorem 3.4(a). Since Q has simple top, each of the modules $Q_1, Q_2, \overline{Q}$ is either indecomposable or 0. If $\overline{Q} \neq 0$, then Q_i is a projective cover of $\overline{Q}$ in $\operatorname{mod} \Lambda/J_i$ for $i = 1, 2$. By assumption, $Q_i \cong \overline{Q}$ for some i, say, $i = 1$. Then $Q \cong Q_2$, i.e., Q is projective over Λ/J_2. If $\overline{Q} = 0$, then $Q \cong Q_1 \oplus Q_2$, whence $Q \cong Q_i$ for some i. Thus (i) holds.

If Y is an indecomposable projective Λ/J_1-module that is not projective over Λ and if P is a projective cover of Y in $\operatorname{mod} \Lambda$, then P is indecomposable so that, according to what we have just proven, P is projective over Λ/J_2. Hence $J_2 P = 0$ and $J_2 Y = 0$ so that $(J_1 + J_2)Y = 0$, i.e., Y is a $\Lambda/(J_1 + J_2)$-module. Since the functor $\operatorname{mod} \Lambda/(J_1 + J_2) \to \operatorname{mod} \Lambda/J_1$ induced by the natural projection $\Lambda/J_1 \to \Lambda/(J_1 + J_2)$ is full, Y is projective over $\Lambda/(J_1 + J_2)$. Thus (ii) holds.

(b) The argument is dual to the proof of (a); we leave it to the reader. ∎

Notation 3.6. Let J_1 and J_2 be two-sided ideals of the left artinian ring Λ for which $J_1 \cap J_2 = 0$, $J_1 + J_2 \neq \Lambda$, and $\Lambda/(J_1 + J_2)$ is a semisimple ring such that the simple projective Λ/J_1-modules coincide with the simple $\Lambda/(J_1 + J_2)$-modules, but no $\Lambda/(J_1 + J_2)$-module is a projective Λ/J_2-module.

PROPOSITION 3.7. *In the setting of Notation* 3.6:

(a) *An indecomposable projective Λ/J_2-module is a projective Λ-module but not a Λ/J_1-module. In particular,* $J_1 \neq 0$.

(b) *No simple $\Lambda/(J_1 + J_2)$-module is a projective Λ-module.*

(c) *The indecomposable hereditary projective Λ/J_2- and Λ-modules coincide.*

(d) *A nonsimple indecomposable projective Λ/J_1-module is a nonhereditary projective Λ-module but not a Λ/J_2-module. In particular, if Λ/J_1 is not semisimple,* $J_2 \neq 0$.

(e) $\mathsf{P} = \mathsf{U} \coprod \mathsf{V}$ *is a disjoint union, where U is a complete set of pairwise nonisomorphic nonsimple indecomposable projective Λ/J_1-modules and V is a complete set of pairwise nonisomorphic indecomposable projective Λ/J_2-modules. If $U \in \mathsf{U}$, $V \in \mathsf{V}$, then* $\operatorname{Hom}_\Lambda(U, V) = 0$ *and, for any* $0 \neq f \in \operatorname{Hom}_\Lambda(V, U)$, $\operatorname{Im} f$ *is a simple $\Lambda/(J_1 + J_2)$-module. In particular, no morphism $V \to U$ is a monomorphism.*

Proof. (a) An indecomposable projective Λ/J_2-module Y is not a $\Lambda/(J_1 + J_2)$-module, whence $J_1 Y \neq 0$. By Proposition 3.5(a), Y is projective over Λ.

(b) A projective cover $Y \to S$ in mod Λ/J_2 of a simple $\Lambda/(J_1 + J_2)$-module S is not an isomorphism by assumption. By (a), Y is a projective cover of S in mod Λ.

(c) By (a), we only have to show that an indecomposable hereditary projective Λ-module P is a Λ/J_2-module. By Proposition 3.5(a), it suffices to show P is not a Λ/J_1-module. If it is, a simple submodule S of P is projective over both Λ and Λ/J_1, hence, a simple $\Lambda/(J_1 + J_2)$-module, in contradiction with (b).

(d) If X is nonsimple indecomposable projective over Λ/J_1, it is not a Λ/J_2-module. It is a nonhereditary projective Λ-module by (c) and Proposition 3.5(a).

(e) By (a), (b), and Proposition 3.5(a), $\mathsf{P} = \mathsf{U} \coprod \mathsf{V}$. Since U is indecomposable projective, if $0 \neq g\colon U \to V$, then $\operatorname{Im} g$ is an indecomposable Λ/J_1-module and a Λ/J_2-module. Hence, Im g is a simple $\Lambda/(J_1 + J_2)$-module and a projective Λ/J_1-module, whence $U \cong \operatorname{Im} g$, which is impossible because U is not simple. Thus $\operatorname{Hom}_\Lambda(U, V) = 0$. If $0 \neq f$: $V \to U$, then, as above, Im f is a simple $\Lambda/(J_1 + J_2)$-module. ∎

PROPOSITION 3.8. *In the setting of Notation* 3.6:

(a) *Let S be a simple $\Lambda/(J_1 + J_2)$-module. If the block of S in Λ/J_1 is not simple, then S is a torsionless Λ-module. If Λ/J_2 is left hereditary, the block of S in Λ/J_1 is not simple if and only if S is a torsionless Λ-module.*

(b) *If some block of Λ/J_1 contains no simple $\Lambda/(J_1 + J_2)$-module, then this block is a block of Λ.*

(c) *If Λ/J_2 is indecomposable and each block of Λ/J_1 contains a simple $\Lambda/(J_1 + J_2)$-module, then Λ is indecomposable.*

Proof. (a) If the block of S in Λ/J_1 is not simple, there is a chain f: $S - X$ in mod Λ/J_1 with $X \not\cong S$ indecomposable projective. Since S is projective, f: $S \to X$. By Proposition 3.7(d), S is a torsionless Λ-module. If Λ/J_2 is left hereditary and S is a submodule of an indecomposable projective Λ-module P, then P is nonhereditary by Proposition 3.7(b) and nonprojective over Λ/J_2 by Proposition 3.7(c). By Proposition 3.7(e), P is a nonsimple indecomposable projective Λ/J_1-module that belongs to the block of S.

(b) If $\{X_1, \ldots, X_s\}$ is a complete set of pairwise nonisomorphic indecomposable projectives in a block of Λ/J_1 that contains no simple $\Lambda/(J_1 + J_2)$-module, each X_i is not simple, hence, projective over Λ by Proposition 3.7(d). We show that if $P \in \mathsf{P}$ satisfies $P \not\cong X_i$ for $i = 1, \ldots, s$, then $\operatorname{Hom}_\Lambda(X_i, P) = 0$ and $\operatorname{Hom}_\Lambda(P, X_i) = 0$. We may assume P is not a Λ/J_1-module; then, by Proposition 3.7(e), P is projective over Λ/J_2, $\operatorname{Hom}_\Lambda(X_i, P) = 0$, and Im f is a simple $\Lambda/(J_1 + J_2)$-module for $0 \neq f$: $P \to X_i$. Since Im f does not belong to the block of X_i in Λ/J_1, we must have $\operatorname{Hom}_\Lambda(P, X_i) = 0$.

(c) If $Q \not\cong P$ and $Q, P \in \mathsf{P}$, we have to construct a chain $Q = Q_0 - Q_1 - \cdots - Q_{s-1} - Q_s = P$ with $Q_j \in \mathsf{P}$. In view of Proposition 3.7(e) and the indecomposability of Λ/J_2, it suffices to show that if Q is a nonsimple projective Λ/J_1-module, the above chain exists for some indecomposable projective Λ/J_2-module P. If S is a simple $\Lambda/(J_1 + J_2)$-module that belongs to the block of Q in Λ/J_1, there exists a chain $Q = Q_0 - \cdots - Q_{s-1} \overset{g_{s-1}}{-} S$, where Q_l is a nonsimple indecomposable projective Λ/J_1-module for $0 \leq l < s$. Since S is projective over Λ/J_1, g_{s-1}: $S \to Q_{s-1}$. Let P be a projective cover of S in mod Λ/J_2, then $Q = Q_0 - \cdots - Q_{s-1} \overset{h}{-} Q_s = P$, where h: $P \to \operatorname{Im} g_{s-1} \cong S$ is a surjective map, is a desired chain. ∎

We now give necessary and sufficient conditions for a pullback to satisfy (A) or be stable.

THEOREM 3.9. *Let Λ be a left artinian ring. In the setting of Notation* 3.6:

(a) Λ *satisfies* (A) *if and only if* Λ/J_1 *is left serial and both* Λ/J_1 *and* Λ/J_2 *satisfy* (A).

For the remaining statements of this theorem, assume Λ *satisfies* (A).

(b) *Nonisomorphic simple* $\Lambda/(J_1 + J_2)$*-modules belong to distinct blocks of* Λ/J_1.

(c) Λ *is indecomposable if and only if* Λ/J_2 *is indecomposable and each block of* Λ/J_1 *contains a simple* $\Lambda/(J_1 + J_2)$*-module.*

(d) Λ *is left serial if and only if* Λ/J_2 *is left serial.*

Proof. (a) If Λ satisfies (A), let X be a nonsimple indecomposable projective Λ/J_k-module, $k = 1$ or 2, let $0 \neq M$ be a nonsimple Λ/J_k-submodule of X, and let Z be an indecomposable direct summand of M. We prove Z is a projective Λ/J_k-module. Proposition 3.7(e) implies that, up to isomorphism, $X \in \mathsf{P}$ so that M, as well as Z, is projective over Λ because Λ satisfies (A). By Proposition 3.7(b), Z is not a simple $\Lambda/(J_1 + J_2)$-module, hence, not a Λ/J_i-module for $i \neq k$, $i \in \{1, 2\}$. By Proposition 3.7(e), Z is a projective Λ/J_k-module so that Λ/J_k satisfies (A). By Propositions 3.7(d) and 2.1(b), Λ/J_1 is left serial.

Suppose Λ/J_1 is left serial and both Λ/J_1 and Λ/J_2 satisfy (A). Let $P \in \mathsf{P}$ and let $0 \neq M$ be a nonsimple submodule of P. We prove M is projective. By Proposition 3.7(e), P is indecomposable projective over Λ/J_k for some $k \in \{1, 2\}$. Since $J_k P = 0$, M is a Λ/J_k-submodule of P which must be projective because Λ/J_k satisfies (A). If $k = 2$, M is projective over Λ by Proposition 3.7(e). If $k = 1$, P is uniserial because Λ/J_1 is left serial. Since Λ/J_1 satisfies (A), M is a nonsimple indecomposable projective Λ/J_1-module, which must be projective over Λ by Proposition 3.7(e).

(b) If, to the contrary, $S = X_0 \overset{f_0}{-} X_1 - \cdots - X_{u-1} \overset{f_{u-1}}{-} X_u = T$ is a chain of nonisomorphic indecomposable projective Λ/J_1-modules, where S and T are simple $\Lambda/(J_1 + J_2)$-modules, we may assume without loss of generality that X_l is not simple for $0 < l < u$. Then X_0 (X_u) is the domain of f_0 (f_{u-1}) and both f_0 and f_{u-1} are monic. Let P (Q) be a projective cover of S (T) in mod Λ/J_2. The chain $P \overset{h}{-} X_1 - \cdots - X_{u-1} \overset{j}{-} Q$, where h (j) is a surjective map of P (Q) onto a submodule of X_1 (X_{u-1}) isomorphic to S (T), contradicts Propositions 3.7(e) and 3.2(a).

(c) By Proposition 3.8(c), only the necessity needs proof. If a block of Λ/J_1 contains no simple $\Lambda/(J_1 + J_2)$-module, Proposition 3.8(b) and the indecomposability of Λ imply that each $P \in \mathsf{P}$ is a Λ/J_1-module. Since

$J_2 \neq \Lambda$, this contradicts Proposition 3.7(a). Thus each block of Λ/J_1 contains a simple $\Lambda(J_1 + J_2)$-module.

If $X \not\cong Y$ are indecomposable projective Λ/J_2-modules, they are projective over Λ by Proposition 3.7(a) and there is a chain $X = Q_0 - Q_1 - \cdots - Q_{u-1} - Q_u = Y$ of nonisomorphic indecomposable projective Λ-modules. By Propositions 3.7(e) and 3.2(a), Q_l is a projective Λ/J_2-module for all l, whence Λ/J_2 is indecomposable.

(d) Follows from (a) and Proposition 3.7(e). ∎

THEOREM 3.10. *Let Λ be a left artinian ring. In the setting of Notation* 3.6, *suppose every simple $\Lambda/(J_1 + J_2)$-module is an injective Λ/J_2-module.*

(a) *Λ is stable if and only if Λ/J_1 is serial and both Λ/J_1 and Λ/J_2 are stable.*

(b) *If Λ is stable, then Λ is serial if and only if Λ/J_2 is serial.*

Proof. (a) If Λ is stable, Λ/J_k satisfies (A) for $k = 1, 2$ by Theorems 2.4 and 3.9(a). To show Λ/J_k is stable, we prove it satisfies (C). Let S be a nonprojective simple torsionless Λ/J_k-module. Then S is not a $\Lambda/(J_1 + J_2)$-module for, otherwise, it is either a projective Λ/J_1-module or an injective Λ/J_2-module, contradicting in either case the assumption that S is not a projective Λ/J_k-module. By Proposition 3.7(e), S is a nonprojective simple torsionless Λ-module. The injective envelope I of S in $\operatorname{Mod} \Lambda/J_k$ is an injective envelope of S in $\operatorname{Mod} \Lambda$ by Proposition 3.5(b). Since Λ is stable, I is uniserial, whence Λ/J_k satisfies (C).

To show Λ/J_1 is serial, we only have to show that every indecomposable injective Λ/J_1-module is uniserial because Λ/J_1 is left serial by Theorem 3.9(a). Since Λ/J_1 satisfies (C), as we have just shown, and since the simple projective Λ/J_1-modules coincide with the simple $\Lambda/(J_1 + J_2)$-modules, we have to show that an injective envelope I of a simple module S in $\operatorname{Mod} \Lambda/J_1$ is uniserial in case S is either a $\Lambda/(J_1 + J_2)$-module or a torsion Λ/J_1-module. In the former case, either $S = I$ if the block of S in Λ/J_1 is simple, or, otherwise, S is a nonprojective torsionless Λ-module by Propositions 3.7(b) and 3.8(a) so that $S \neq E$, where E is the injective envelope of S in $\operatorname{Mod} \Lambda$. Since S is an injective Λ/J_2-module, $I \cong E$ by Proposition 3.5(b). Since E is uniserial because Λ is stable, so is I. If S is a torsion Λ/J_1-module, it is not a Λ/J_2-module, hence it is a torsion Λ-module by Proposition 3.7(e). A projective cover Q of S in $\operatorname{mod} \Lambda/J_1$ is a nonhereditary projective Λ-module by Proposition 3.7(d). An injective envelope $I(Q)$ of Q in $\operatorname{Mod} \Lambda$ is a projective uniserial Λ-module by Proposition 2.1(b) and (B), and is not a Λ/J_2-module because neither is S. Hence, $I(Q)$ is a Λ/J_1-module by Proposition 3.7(e) so that $I(Q)/\mathbf{r}Q$ is a torsion uniserial injective Λ- and Λ/J_1-module by Proposition 2.16. Since $\operatorname{Soc} I(Q)/\mathbf{r}Q \cong S$, then $I \cong I(Q)/\mathbf{r}Q$.

For the sufficiency, suppose Λ/J_1 is serial and both Λ/J_1 and Λ/J_2 are stable. Since Λ satisfies (A) by Theorem 3.9(a), it is enough to prove Λ satisfies (C). Let I be an injective envelope in Mod Λ of a nonprojective simple submodule S of an indecomposable projective Λ-module P. By Proposition 3.7(e), P is a nonsimple indecomposable projective Λ/J_k-module and S is not injective over Λ/J_k, so that the injective envelope E of S in Mod Λ/J_k is not simple. By Proposition 3.5(b), E is injective in Mod Λ, i.e., $I \cong E$. If $k = 1$, E is uniserial because Λ/J_1 is serial. If $k = 2$, S is not projective over Λ/J_2 by Proposition 3.7(c), whence E is uniserial by (C).

(b) Suppose Λ is stable. By Theorem 3.9(d), we only have to show that all nonsimple indecomposable injective Λ-modules are uniserial if and only if all nonsimple indecomposable injective Λ/J_2-modules are uniserial. Since Λ/J_1 is serial by (a), the statement is a consequence of Proposition 3.5(b). ∎

Notation 3.11. For the left artinian ring Λ, denote by

$$\mathbf{S} = \{S_1, \dots, S_m\}, \qquad m \geq 0,$$

a complete set of pairwise nonisomorphic nonprojective simple torsionless Λ-modules whose projective covers are hereditary projective, and fix a decomposition

$$\mathbf{P} = \mathbf{P}' \oplus \mathbf{P}''$$

of $\mathbf{P}$ as a direct sum of left ideals of Λ, where every indecomposable direct summand of $\mathbf{P}'$ is a projective cover of S_i for some i and, for all i, no indecomposable direct summand of $\mathbf{P}''$ is a projective cover of S_i. Put

$$I_1 = \mathbf{r}\mathbf{P}' + \mathbf{P}'' \quad \text{and} \quad I_2 = \mathbf{Q}.$$

We now assume that Λ satisfies (A) or is stable and show how to construct a pair of ideals satisfying the hypotheses of Theorem 3.9 or 3.10, respectively.

PROPOSITION 3.12. *In the setting of Notation* 3.11, *assume* Λ *is an indecomposable left artinian ring satisfying* (A), $\mathbf{P} \neq 0$, *and* $\mathbf{Q} \neq 0$.

(a) $\mathbf{S} \neq \varnothing$ *and, for* $i = 1, \dots, m$, S_i *is not injective*.

(b) I_1 *and* I_2 *are two-sided ideals of* Λ *satisfying* $I_1 \cap I_2 = 0$ *and* $I_1 + I_2 \neq \Lambda$.

(c) $\Lambda/(I_1 + I_2)$ *is a semisimple ring and* $\mathbf{S}$ *is a complete set of pairwise nonisomorphic simple* $\Lambda/(I_1 + I_2)$*-modules*.

(d) $\mathbf{P}$ *is a* Λ/I_2*-module isomorphic to* Λ/I_2. $\mathbf{Q}$ *is a* Λ/I_1*-module, and there exists an isomorphism* $\Lambda/I_1 \cong (\oplus_{i=1}^m S_i^{n_i}) \oplus \mathbf{Q}$ *in* mod Λ/I_1 *for some positive integers* n_i.

(e) *$\mathcal{S}$ is a complete set of pairwise nonisomorphic simple projective Λ/I_1-modules, and for $i = 1, \ldots, m$, S_i is not projective as a Λ/I_2-module.*

(f) *If Λ is stable, S_i is an injective Λ/I_2-module, $i = 1, \ldots, m$.*

Proof. (a) Since Λ is indecomposable and $\mathbf{P} \neq 0$, $\mathbf{Q} \neq 0$, there exists a chain $P - Q$ of indecomposable projective Λ-modules with P hereditary and Q nonhereditary. By Corollary 2.2(b) and (c), $P \to Q$ and P is a projective cover of the nonprojective simple Λ-module $\operatorname{Soc} Q$, so that $\mathcal{S} \neq \varnothing$. Since an injective torsionless module must be projective, S_i is not injective.

(b) To show I_1 and I_2 are two-sided ideals, we prove that $f(I_j) \subset I_j$ for all $f \in \operatorname{End}_\Lambda(\Lambda)$, $j = 1, 2$. Since Hom is an additive functor, Corollary 2.2(b) implies $\operatorname{Hom}_\Lambda(I_2, \mathbf{P}) = 0$ so that $f(I_2) \subset I_2$. We have $f(I_1) = f(\mathbf{r}\mathbf{P}' + \mathbf{P}'') = \mathbf{r}f(\mathbf{P}') + f(\mathbf{P}'')$. Since no indecomposable summand of $\mathbf{P}''$ is a projective cover of a nonprojective simple torsionless Λ-module, Corollary 2.2(c) implies $\operatorname{Hom}_\Lambda(\mathbf{P}'', \mathbf{Q}) = 0$. Then $f(\mathbf{P}'') \subset \mathbf{P}$, whence $f(\mathbf{P}'') \subset \mathbf{r}\mathbf{P}' + \mathbf{P}'' = I_1$ because $\mathbf{P}'$ and $\mathbf{P}''$ have no isomorphic nonzero direct summands. By a similar argument, taking into account Corollary 2.2(c), we get $f(\mathbf{P}') \subset \mathbf{P}' + \mathbf{r}\mathbf{P}'' + \operatorname{Soc}\mathbf{Q}$, whence $\mathbf{r}f(\mathbf{P}') \subset \mathbf{r}\mathbf{P}' + \mathbf{r}^2\mathbf{P}'' + \mathbf{r}\operatorname{Soc}\mathbf{Q} \subset I_1$. Thus $f(I_1) \subset I_1$.

It is obvious that $I_1 \cap I_2 = 0$. By (a), $\mathbf{P}' \neq 0$ so that $\mathbf{r}\mathbf{P}' \neq \mathbf{P}'$ by Nakayama's lemma, whence $I_1 \subsetneq \mathbf{P}$ and $I_1 + I_2 \neq \Lambda$.

(c) Follows immediately from (b).

(d) Obviously, $\Lambda/I_2 \cong \mathbf{P}$. By (c), S_i is a Λ/I_1-module for all i. By (b), $I_1 \cap I_2 = 0$ so that $I_1 I_2 = 0$. Therefore, $\mathbf{Q}$ is a Λ/I_1-module and $\Lambda/I_1 \cong (\oplus_{i=1}^m S_i^{n_i}) \oplus \mathbf{Q}$ in $\operatorname{mod}(\Lambda/I_1)$ for some positive integers n_i.

(e) By (d), the indecomposable projective Λ/I_1-modules (up to isomorphism) are the S_is and the indecomposable nonhereditary projective Λ-modules, the latter not being simple.

(f) Denote by E an injective envelope of S_i in $\operatorname{mod}\Lambda/I_2$ and by I, an injective envelope of S_i in $\operatorname{mod}\Lambda$; here E is isomorphic to the submodule of I annihilated by I_2. Since Λ is stable, it satisfies (B) by Theorem 2.4 so that I is an indecomposable nonhereditary projective Λ-module. By Proposition 2.1(b), if $Z \neq S_i$ is a nonzero submodule of I, then Z is an indecomposable nonhereditary projective Λ-module, hence, a direct summand of $\mathbf{Q} = I_2$. By (d), $I_2\mathbf{P} = 0$ so that $I_2^2 = I_2$ whence $I_2 Z \neq 0$. It follows that $E \cong S_i$. ∎

COROLLARY 3.13. *In the setting of Notation* 3.11, *assume Λ is an indecomposable left artinian ring satisfying* (A), $\mathbf{P} \neq 0$, *and* $\mathbf{Q} \neq 0$.

(a) *Λ/I_2 is an indecomposable left hereditary ring and S_i is a torsion Λ/I_2-module, $i = 1, \ldots, m$.*

(b) Λ/I_1 *is a direct product of* m *rings* $\Gamma_1, \dots, \Gamma_m$, *where* Γ_i *is a nonsemisimple indecomposable left serial ring satisfying* (A) *and* S_i *is a unique* (*up to isomorphism*) *simple projective* Γ_i*-module*, $i = 1, \dots, m$. *If* Λ *is stable,* Γ_i *is serial stable for all* i; *if* Λ *is serial stable,* $m = 1$, *i.e.,* Λ/I_1 *is indecomposable*.

Proof. (a) By Propositions 3.12(d) and 3.7(c), every indecomposable projective Λ/I_2-module is hereditary so that Λ/I_2 is left hereditary. By Proposition 3.12 and Theorem 3.9(c), Λ/I_2 is indecomposable. By Proposition 3.12(e), S_i is a nonprojective simple Λ/I_2-module, hence, a torsion module.

(b) Proposition 3.12 and Theorem 3.9(b) and (c) imply that each block of Λ/I_1 contains precisely one of the S_is; we denote by Γ_i the block of Λ/I_1 containing S_i. By Proposition 3.12(e), S_i is a unique (up to isomorphism) simple projective Γ_i-module. Since Λ/I_2 is left hereditary by (a), Proposition 3.8(a) implies that Γ_i is not semisimple.

If Λ is stable, Λ/I_1 is serial by Proposition 3.12 and Theorem 3.10(a), whence Γ_i is serial for all i. If Λ is serial stable, then by (a) and Theorem 3.10(b), Λ/I_2 is an indecomposable serial left hereditary ring. Hence, Λ/I_2 has a unique up to isomorphism simple injective module and $m = 1$ by Proposition 3.12(f). ∎

We now prove that if Λ satisfies (A), then Proposition 3.12 and Corollary 3.13(a) give a unique representation of Λ as a pullback of a left hereditary ring and a left serial ring satisfying (A) over a semisimple ring.

THEOREM 3.14. *In the setting of Notation* 3.11, *assume* Λ *is an indecomposable left artinian ring satisfying* (A), $\mathbf{P} \neq 0$, *and* $\mathbf{Q} \neq 0$. *Let* J_1 *and* J_2 *be two-sided ideals of* Λ *satisfying the conditions of Notation* 3.6 *and such that each simple* $\Lambda/(J_1 + J_2)$*-module is a torsion* Λ/J_2*-module. Then the following are equivalent*:

(a) $J_1 = I_1$ *and* $J_2 = I_2$.

(b) $J_1 + J_2 = \operatorname{ann}_\Lambda \mathcal{S}$.

(c) Λ/J_1 *has no simple blocks and* Λ/J_2 *is a left hereditary ring*.

Proof. (a) ⇒ (b) Follows from Proposition 3.12(c).

(b) ⇒ (c) By Theorem 3.9(a), Λ/J_2 satisfies (A). If, to the contrary, Λ/J_2 is not left hereditary, there exists an indecomposable nonhereditary projective Λ/J_2-module and, since $\mathbf{P} \neq 0$, an indecomposable hereditary projective Λ- and Λ/J_2-module by Proposition 3.7(c). Since Λ/J_2 is indecomposable by Theorem 3.9(c), there is a chain $Q \overset{f}{-} P$ of indecomposable Λ/J_2-projectives with Q nonhereditary and P hereditary, whence $\operatorname{Soc} Q$ is simple by Proposition 2.1(b), and f: $P \to \operatorname{Soc} Q$ is a projective cover by Corollary 2.2(b) and (c). By Proposition 3.7(a), $\operatorname{Soc} Q \in \mathsf{S}$ so that, by (b), $\operatorname{Soc} Q$ is a simple $\Lambda/(J_1 + J_2)$-module, hence, a torsion Λ/J_2-module, a contradiction.

We have shown that Λ/J_2 is a left hereditary ring. By Proposition 3.8(a), in order to prove that Λ/J_1 has no simple blocks, it suffices to show that each simple $\Lambda/(J_1 + J_2)$-module S is a torsionless Λ-module. The latter holds because $J_1 + J_2 = \operatorname{ann}_\Lambda \mathsf{S}$ implies $S \in \mathsf{S}$.

(c) $\Rightarrow$ (a) Since Λ/J_2 is left hereditary, Propositions 3.12(d) and 3.7(c) imply that the Λ-modules Λ/I_2 and Λ/J_2 have the same indecomposable direct summands, namely, the nonisomorphic indecomposable hereditary projective Λ-modules. Hence $I_2 = \operatorname{ann}_\Lambda \Lambda/I_2 = \operatorname{ann}_\Lambda \Lambda/J_2 = J_2$. By Propositions 3.12(d) and 3.7(d) and (e), the Λ-modules Λ/I_1 and Λ/J_1 have the same nonsimple indecomposable direct summands, namely, the nonisomorphic indecomposable nonhereditary projective Λ-modules. We show that Λ/I_1 and Λ/J_1 have the same simple direct summands.

Let S be a simple direct summand of Λ/I_1. Since $S \in \mathsf{S}$ by Proposition 3.12(d), its projective cover P in $\operatorname{mod} \Lambda$ is indecomposable hereditary and S is a submodule of an indecomposable nonhereditary projective Λ-module Q. Since Λ/J_2 is left hereditary, Q is a Λ/J_1-module and P is a Λ/J_2-module by Proposition 3.7(c) and (e). Then S is both a Λ/J_1- and Λ/J_2-module, hence, a simple $\Lambda/(J_1 + J_2)$-module and a projective Λ/J_1-module. Thus S is a direct summand of Λ/J_1.

If T is a simple direct summand of Λ/J_1, then T is a $\Lambda/(J_1 + J_2)$-module and, since Λ/J_2 is left hereditary, the projective cover P of T in $\operatorname{mod} \Lambda/J_2$ is an hereditary projective Λ-module by Proposition 3.7(c). Since Λ/J_1 has no simple blocks, T is a nonprojective torsionless Λ-module by Propositions 3.7(b) and 3.8(a). Hence, $T \in \mathsf{S}$ and T is a direct summand of Λ/I_1 by Proposition 3.12(d).

Since the Λ-modules Λ/I_1 and Λ/J_1 have the same indecomposable direct summands, $I_1 = \operatorname{ann}_\Lambda \Lambda/I_1 = \operatorname{ann}_\Lambda \Lambda/J_1 = J_1$. ∎

COROLLARY 3.15. *Let Λ be an indecomposable left artinian ring with* $\mathbf{P} \neq 0$, $\mathbf{Q} \neq 0$.

(a) Λ *satisfies* (A) (*is stable*) *if and only if there exist an indecomposable left hereditary left artinian ring Δ with a fixed collection $\mathsf{T} = \{T_1, \ldots, T_r\}$, $r > 0$, of pairwise nonisomorphic nonprojective* (*injective nonprojective*) *simple Δ-modules and indecomposable nonsemisimple left serial* (*serial*) *rings $\Theta_1, \ldots, \Theta_r$, each satisfying* (A) (*stable*) *and having an indecomposable heredi-*

tary projective module, subject to the following two conditions:

(i) $\Delta/\operatorname{ann}_\Delta T_i \cong \Theta_i/\operatorname{ann}_{\Theta_i} U_i$, *where* U_i *is the unique up to isomorphism simple projective* Θ_i*-module*.

(ii) *There exists a pullback diagram of rings*

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \prod_{i=1}^{r} \Theta_i \arrow[d, "g"] \\ \Delta \arrow[r, "h"] & \Delta\Big/\bigcap_{i=1}^{r} \operatorname{ann}_\Delta T_i \cong \prod_{i=1}^{r} \Theta_i/\operatorname{ann}_{\Theta_i} U_i, \end{tikzcd}$$

where g and h are the natural projections.

(b) *Suppose conditions* (i) *and* (ii) *of part* (a) *are satisfied. A complete set of pairwise nonisomorphic simple projective* Δ*-modules is a complete set of pairwise nonisomorphic simple projective* Λ*-modules. If each* Θ_i *is serial stable, denote by* V_i *the unique up to isomorphism simple injective* Θ_i*-module*; *then* $\{V_1, \dots, V_r\}$ *is a collection of pairwise nonisomorphic simple injective* Λ*-modules*.

(c) *Let* Δ' *be an indecomposable left hereditary left artinian ring with a fixed collection* $\mathcal{T}' = \{T'_1, \dots, T'_r\}$, $r' > 0$, *of pairwise nonisomorphic nonprojective simple* Δ'*-modules, and let* $\Theta'_1, \dots, \Theta'_{r'}$, *be indecomposable nonsemisimple left serial rings, each satisfying* (A) *and having an indecomposable hereditary projective module. Denote by* U'_i *the unique up to isomorphism simple projective* Θ'_i*-module. If* Δ' *and the* Θ'_i*s satisfy conditions* (i) *and* (ii) *of part* (a), *then* $r' = r$ *and there exist a permutation* σ *of the set* $\{1, \dots, r\}$ *and isomorphisms of rings* $f\colon \Delta \to \Delta'$, $f_i\colon \Theta_i \to \Theta'_{\sigma(i)}$ *satisfying* $f^*(T'_{\sigma(i)}) \cong T_i$, $f_i^*(U'_{\sigma(i)}) \cong U_i$, $i = 1, \dots, r$, *where* $f^*\colon \operatorname{mod}\Delta' \to \operatorname{mod}\Delta$, $f_i^*\colon \operatorname{mod}\Theta'_{\sigma(i)} \to \operatorname{mod}\Theta_i$ *are the functors induced by* f, f_i, *respectively*.

Proof. (a) Necessity follows from Proposition 3.12 and Corollary 3.13 by choosing $\Delta = \Lambda/I_2$, $r = m$, and, for $i = 1, \dots, m$, $\Theta_i = \Gamma_i$ and $T_i = U_i = S_i$, where I_2 and the Γ_is are defined in Notation 3.11 and Corollary 3.13, respectively. For sufficiency, note that, for each i, Θ_i has a unique (up to isomorphism) simple projective module by Proposition 3.2(c) and put $\Lambda_1 = \prod_{i=1}^{r} \Theta_i$, $\Lambda_2 = \Delta$, and $\overline{\Lambda} = \Delta/\bigcap_{i=1}^{r} \operatorname{ann}_\Delta T_i \cong \prod_{i=1}^{r} \Theta_i/\operatorname{ann}_{\Theta_i} U_i$, where the above isomorphism of rings is chosen so that $T_i \cong U_i$ as simple $\overline{\Lambda}$-modules. Denote by J_1, J_2 two-sided ideals of Λ satisfying $J_1 \cap J_2 = 0$ and having the property that the pullback diagram (3.2) of rings is isomorphic to the pullback diagram (3.1) given by condition (ii) (both diagrams are drawn before Theorem 3.4). Since the ring $\overline{\Lambda}$ is semisimple and each of the finite collections $\mathcal{T} = \{T_1, \dots, T_r\}$ and $\mathcal{U} = \{U_1, \dots, U_r\}$ is a complete set of pairwise nonisomorphic simple $\overline{\Lambda}$-modules, the simple projective Λ_1-modules coincide with the simple $\overline{\Lambda}$-modules and, by assumption, no simple $\overline{\Lambda}$-module is a projective Λ_2-module. By Theorem 3.9(a) applied to the pullback diagram (3.2), Λ satisfies (A). If each T_i is an injective Δ-module and each Θ_i is serial stable, Λ is stable by Theorem 3.10(a).

(b) Follows from Propositions 3.7(c) and 3.5(b).

(c) Put $\Lambda'_1 = \prod_{i=1}^{r'} \Theta'_i$, $\Lambda'_2 = \Delta'$, and $\overline{\Lambda}' = \Delta' / \bigcap_{i=1}^{r'} \operatorname{ann}_{\Delta'} T'_i \cong \prod_{i=1}^{r'} \Theta'_i / \operatorname{ann}_{\Theta'_i} U'_i$, where U'_i is the unique up to isomorphism simple projective Θ'_i-module and the above isomorphism of rings is chosen so that $T'_i \cong U'_i$ as simple $\overline{\Lambda}'$-modules. By Theorem 3.14, pullback diagram

$$\begin{array}{ccc} \Lambda' & \longrightarrow & \Lambda'_1 \\ \downarrow & & \downarrow \\ \Lambda'_2 & \longrightarrow & \overline{\Lambda}' \end{array} \tag{3.4}$$

given by condition (ii) is isomorphic to pullback diagram (3.2), so that diagrams (3.1) and (3.4) are isomorphic. Let $\phi\colon \Lambda_1 \to \Lambda'_1$, $f\colon \Lambda_2 \to \Lambda'_2$, and $\bar{f}\colon \overline{\Lambda} \to \overline{\Lambda}'$ be isomorphisms of rings coming from the isomorphism of pullback diagrams (3.1) and (3.4).

Since $\{\Theta_1, \dots, \Theta_r\}$ and $\{\Theta'_1, \dots, \Theta'_{r'}\}$ are complete sets of indecomposable blocks for isomorphic rings Λ_1 and Λ'_1, respectively, we conclude that $r' = r$ and there exists a permutation σ of the set $\{1, \dots, r\}$ such that the isomorphism ϕ can be viewed as a collection of ring isomorphisms $f_i\colon \Theta_i \to \Theta'_{\sigma(i)}$. Since U_i ($U'_{\sigma(i)}$) is the unique up to isomorphism simple projective Θ_i- ($\Theta'_{\sigma(i)}$-)module, $f_i^*(U'_{\sigma(i)}) \cong U_i$ for $i = 1, \dots, r$, where $f_i^*\colon \operatorname{mod} \Theta'_{\sigma(i)} \to \operatorname{mod} \Theta_i$ is the isomorphism of categories induced by the isomorphism of rings f_i. The commutative diagrams of rings with surjective vertical morphisms

$$\begin{array}{ccc} \Lambda_1 & \xrightarrow{\phi} & \Lambda'_1 \\ \downarrow & & \downarrow \\ \overline{\Lambda} & \xrightarrow{\bar{f}} & \overline{\Lambda}' \end{array} \quad \text{and} \quad \begin{array}{ccc} \Lambda_2 & \xrightarrow{f} & \Lambda'_2 \\ \downarrow & & \downarrow \\ \overline{\Lambda} & \xrightarrow{\bar{f}} & \overline{\Lambda}' \end{array}$$

give rise to the commutative diagrams

$$\begin{array}{ccc} \operatorname{mod} \overline{\Lambda}' & \xrightarrow{\bar{f}^*} & \operatorname{mod} \overline{\Lambda} \\ \downarrow & & \downarrow \\ \operatorname{mod} \Lambda'_1 & \xrightarrow{\phi^*} & \operatorname{mod} \Lambda_1 \end{array} \quad \text{and} \quad \begin{array}{ccc} \operatorname{mod} \overline{\Lambda}' & \xrightarrow{\bar{f}^*} & \operatorname{mod} \overline{\Lambda} \\ \downarrow & & \downarrow \\ \operatorname{mod} \Lambda'_2 & \xrightarrow{f^*} & \operatorname{mod} \Lambda_2, \end{array}$$

respectively, of categories and functors with vertical functors full embeddings. Since $f_i^*(U'_{\sigma(i)}) \cong U_i$ and $T_i \cong U_i$ $(T'_{\sigma(i)} \cong U'_{\sigma(i)})$ as $\overline{\Lambda}$- $(\overline{\Lambda}'$-)modules, $f^*(T'_{\sigma(i)}) \cong T_i$. ∎

We give an example of how to use the above corollary to construct a stable ring from a left hereditary ring and a serial left hereditary ring.

EXAMPLE 3.16. The ring $\Delta = \left(\begin{smallmatrix} \mathbb{R} & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right)$ is an hereditary finite-dimensional algebra over $\mathbb{R}$. The unique simple injective Δ-module is of the form $T = P/\mathbf{r}P$, where $P = \left(\begin{smallmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right)$. Note that $\operatorname{ann}_\Delta T = \left(\begin{smallmatrix} \mathbb{R} & \mathbb{C} \\ 0 & 0 \end{smallmatrix}\right)$ and $\Delta/\operatorname{ann}_\Delta T \cong \mathbb{C}$. The ring $\Theta = \left(\begin{smallmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right)$ is a serial hereditary finite-dimensional algebra over $\mathbb{C}$, $U = \left(\begin{smallmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is the unique simple projective Θ-module, $\operatorname{ann}_\Theta U = \left(\begin{smallmatrix} 0 & \mathbb{C} \\ 0 & \mathbb{C} \end{smallmatrix}\right)$, and $\Theta/\operatorname{ann}_\Theta U \cong \mathbb{C}$.

Consider the pullback diagram of rings

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \Theta \arrow[d, "g"] \\ \Delta \arrow[r, "h"] & \mathbb{C}, \end{tikzcd}$$

where g and h are the natural projections. It is easy to check that $\Lambda \cong \Gamma/(\operatorname{rad}\Gamma)^2$, where

$$\Gamma = \begin{pmatrix} \mathbb{R} & \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} & \mathbb{C} \\ 0 & 0 & \mathbb{C} \end{pmatrix}.$$

By the above corollary, Λ is stable.

COROLLARY 3.17. *Let Λ be an indecomposable ring with $\mathbf{P} \neq 0$, $\mathbf{Q} \neq 0$.*

(a) *Λ is left serial satisfying* (A) (*serial stable*) *if and only if there exist indecomposable nonsemisimple left serial* (*serial*) *left hereditary rings $\Delta_0, \Delta_1, \ldots, \Delta_t$, $t > 0$, such that for a nonprojective* (*the unique up to isomorphism injective nonprojective*) *simple Δ_i-module V_i, where $i = 1, \ldots, t$, the following two conditions hold*:

(i) $\Delta_i/\operatorname{ann}_{\Delta_i} V_i \cong \Delta_{i-1}/\operatorname{ann}_{\Delta_{i-1}} U_{i-1}$, *where for $j = 0, 1, \ldots, t$, U_j is the unique up to isomorphism simple projective Δ_j-module.*

(ii) *Let $\Sigma_0, \Sigma_1, \ldots, \Sigma_t$ be a sequence of rings with the property that $\Sigma_0 = \Delta_0$ and, for $0 < i \le t$,*

$$\begin{tikzcd} \Sigma_i \arrow[rr] \arrow[d] & & \Sigma_{i-1} \arrow[d, "g_i"] \\ \Delta_i \arrow[r, "h_i"] & \Delta_i/\operatorname{ann}_{\Delta_i} V_i \cong & \Sigma_{i-1}/\operatorname{ann}_{\Sigma_{i-1}} U_{i-1} \end{tikzcd}$$

is a pullback diagram of rings, where g_i *and* h_i *are the natural projections. Then* $\Sigma_t \cong \Lambda$.

(b) *Suppose conditions* (i) *and* (ii) *of part* (a) *are satisfied. For* $i = 1, \dots, t$, U_i *is the unique up to isomorphism simple projective* Σ_i*-module; if each* Δ_i *is serial, the unique up to isomorphism simple injective* Δ_0*-module* V_0 *is the unique up to isomorphism simple injective* Σ_i*-module.*

(c) *Let* $\Delta'_0, \Delta'_1, \dots, \Delta'_{t'}$, $t' > 0$, *be indecomposable nonsemisimple left serial left hereditary rings, and let* V'_i *be a nonprojective simple* Δ'_i*-module,* $i = 1, \dots, t'$. *If the rings* Δ'_i *and modules* V'_i *satisfy conditions* (i) *and* (ii) *of part* (a), *then* $t' = t$ *and there exist isomorphisms of rings* $f_i\colon \Delta_i \to \Delta'_i$ *satisfying* $f_i^*(V'_i) \cong V_i$, *where* $f_i^*\colon \operatorname{mod} \Delta'_i \to \operatorname{mod} \Delta_i$ *is the functor induced by* f_i, $i = 0, 1, \dots, t$.

Proof. (a) Follows from Corollary 3.15(a) and Theorems 3.9(a) and (d) and 3.10(a) and (b) by induction on t.

(b) Follows from Corollary 3.15(b).

(c) Follows from Corollary 3.15(c) by induction on t. ∎

ACKNOWLEDGMENTS

The authors are grateful for the helpful remarks of the referee.

REFERENCES

[AF] F. W. Anderson and K. R. Fuller, "Rings and Categories of Modules," 2nd ed., Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, New York, 1991.

[A] M. Auslander, Functors and morphisms determined by objects, *in* "Representation Theory of Algebras, Proceedings of the Philadelphia Conference," Lecture Notes in Pure and Applied Mathematics, Vol. 37, Dekker, New York, 1978.

[AR] M. Auslander and I. Reiten, Stable equivalence of artin algebras, *in* "Proceedings of the Conference on Orders, Group Rings and Related Topics, Ohio State University, 1972," Lecture Notes in Mathematics, Vol. 353, pp. 8–64, Springer-Verlag, Berlin/New York, 1973.

[ARS] M. Auslander, I. Reiten and S. O. Smalø, "Representation Theory of Artin Algebras," Cambridge Studies in Advanced Mathematics, Vol. 36, Cambridge Univ. Press, New York, 1994.

[B] P. Brown, Global dimension of quasihereditary serial rings, *Comm. Algebra* **22**(4) (1994), 1271–1280.

[DP] R. F. Damiano and Z. Papp, On consequences of stability, *Comm. Algebra* **9**(7) (1981), 747–764.

[FV] A. Facchini and P. Vamos, Injective modules over pullbacks, *J. London Math. Soc. (2)* **31** (1985), 425–438.

[GS] R. Gordon and L. W. Small, Piecewise domains, *J. Algebra* **23** (1972), 553–564.
[J] S. Jagadeeshan, "Stable Artin Algebras," Ph.D. thesis, Syracuse University, 1994.
[JK1] S. Jagadeeshan and M. Kleiner, Stable artin algebras: The transpose and the global dimension, *in* "Seventh Conference on Representations of Algebras (ICRA VII)," Canadian Mathematical Society Conference Proceedings, Vol. 18, pp. 343–351, 1996.
[JK2] S. Jagadeeshan and M. Kleiner, Projectively stable artin algebras: Representation theory and quivers with relations, preprint.
[L] L. S. Levy, Modules over pullbacks and subdirect sums, *J. Algebra* **71** (1981), 50–61.
[M] J. W. Milnor, "Introduction to Algebraic K-Theory," Annals of Mathematics Studies, Vol. 72, Princeton Univ. Press, Princeton, NJ, 1971.
[P] Z. Papp, On stable noetherian rings, *Trans. Amer. Math. Soc.* **213** (1975), 107–114.
[Pl] M. I. Platzeck, Representation theory of algebras stably equivalent to an hereditary artin algebra, *Trans. Amer. Math. Soc.* **238** (1978), 89–128.
[SV] J. D. Sally and W. V. Vasconcelos, Stable rings and a problem of Bass, *Bull. Amer. Math. Soc.* **79** (1973), 574–576.